1. Introductory part

In order to follow the contents of this book with full understanding, certain prerequisites from high school mathematics will be necessary. This firstly pertains to the basic concepts from the plane geometry and the space geometry. From the field of planimetry one should be familiar with all the facts and theorems about the triangle that are included in the standard high school program. The same is valid for spatial geometry, where it is required to have a solid knowledge on the mutual positions of lines and planes in space, with special emphasis on parallelism and perpendicularity.

In the proofs of theorems and the solutions of problems we tried to use the geometrical (synthetic) method wherever possible. Sometimes, however, we were not able to evade the usage of trigonometry or, in certain cases, the very nature of a problem or a theorem required such an approach. Therefore it is assumed that the reader has mastered the basic concepts of trigonometrical functions and trigonometrical facts about the triangle.

In the same way, a necessity of vector algebra quite naturally occurs in the text. Hence, the reader should be familiar with the concepts of vector addition and subtraction, as well as with multiplication of a vector by a scalar, furthermore with linear independency of vectors, inner product, vector product and mixed product of vectors. In several problems, a tetrahedron is given by the coordinates of its vertices in a coordinate system in space. That requires elementary knowledge of analytical geometry of space, such as distance between two points in space, equation of a plane, distance between a point and a plane and so on.

We also use some theorems and formulas of elementary mathematics which either rarely occur or do not occur at all in high school programs in mathematics.

Such theorems and formulas will be explicitly stated and some of the proofs will be given, too.

There is a well-known **Heron's formula for the area of a triangle** with given sidelengths a, b and c. That formula reads $P = \sqrt{s(s-a)(s-b)(s-c)}$, where s denotes the semiperimeter of the triangle. This formula may be written in another form:

$$P = \frac{1}{4}\sqrt{(a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4)}. \qquad \text{(F1)}$$

The formula (F1) is suitable if sidelengths are real numbers or algebraic expressions under the square root sign $\sqrt{}$, which is the reason why it will be repeatedly used in such cases. The proof of the formula may be found in [2].

We are going to point out three particular theorems for the triangle. The first reason for that is that these theorems are used in some proofs, and the second reason is that there exist analogous theorems for the tetrahedron, more of which will be said later on in the book.

Th1. *Let P, Q, R be given points on the respective sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of a triangle ABC. The lines AP, BQ and CR intersect in one point if and only if $|AR|\cdot|BP|\cdot|CQ| = |BR|\cdot|CP|\cdot|AQ|$ (Fig. 1.1.).*

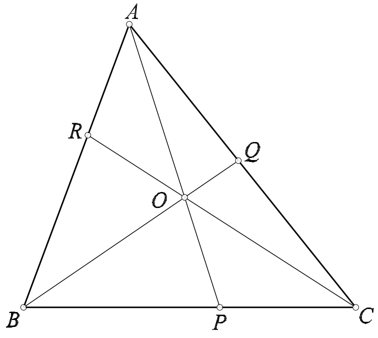

Fig. 1.1.

This is **Ceva's theorem**. It was stated and proved by the Italian mathematician **Giovanni Ceva** (1648. – 1734.).

Th2. *If a line intersects the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of a triangle ABC in respective points P, Q and R, then it holds: $|AR|\cdot|BP|\cdot|CQ| = |BR|\cdot|CP|\cdot|QA|$ (Fig. 1.2.).*

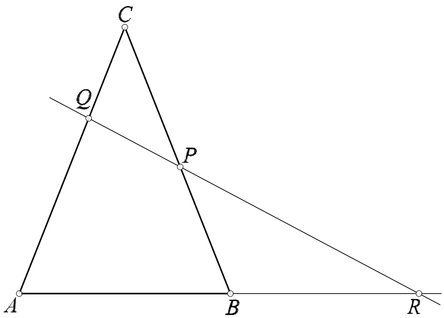

Fig. 1.2.

This theorem is named after the ancient Greek mathematician **Menelaus**. The following one is **Van Aubel's theorem**.

Th. *If P, Q, R are points on the respective sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of a triangle ABC chosen in such a way that the lines AP, BQ and CR intersect at a point O, then it holds $\dfrac{|CO|}{|OR|} = \dfrac{|CQ|}{|QA|} + \dfrac{|CP|}{|PB|}$ (Fig. 1.1.).*

Let us observe certain connections between the above theorems. Ceva's and Menelaus' theorem are so-called dual theorems and are expressed by the same formula. Furthermore, the assumptions on the triangle in both Ceva's and Van Aubel's theorem are identical. Proofs of these theorems may be found in [2]. (In that book, several proofs are supplied for each theorem. In the case of Ceva's theorem, for instance, seven different proofs are given). There is also a certain less known theorem on the orthocentre of a triangle, which is used in the book:

Th4. *The orthocentre of a triangle divides each of its altitudes into two segments whose product is constant and equal to $4R^2 \cos\alpha \cos\beta \cos\gamma$, where R is the circumradius and α, β, γ the angles of the triangle.*

The proof is given in [2].

A theorem concerning a particular position of lines in space is known as the **Theorem of three perpendiculars** and reads as follows:

Th5. *Let a and b be two mutually perpendicular lines of a plane π, let A be their intersection point and let c be the line perpendicular to the plane π, passing through a point O of the line a, with $O \neq A$.*

If C is any point on c, then the lines CA and b are perpendicular, too.

Proof.

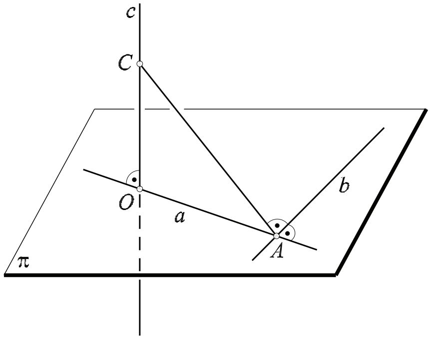

Fig. 1.3.

Since the line c is perpendicular to the plane π, c is also perpendicular to any line in that plane, including the line b (Fig. 1.3.). We see that the line b is perpendicular to the lines a and c. Therefore, b is perpendicular to any line in the plane determined by a and c, hence perpendicular to the line CA, too, which is the assertion of the theorem. Q.E.D.

Next theorem will be frequently used in the proofs of some theorems, as well as in the solutions of several problems. Therefore, we are going to prove it. The theorem reads as follows:

Th6. *Let φ be the angle between the planes $\sum_1$ and $\sum_2$. If a polygon of area P is situated in the plane $\sum_1$ and the orthogonal projection of that polygon to the plane $\sum_2$ is a polygon of area Q, then $Q = P\cos\varphi$.*

Proof. We use the notation as in fig. 1.4 Every polygon may be decomposed into a finite number of triangles. The area of a triangle does not depend on its position in the plane $\sum_1$, so that the same holds for its orthogonal projection. Therefore it suffices to prove that the theorem is valid for a triangle in such a position as shown in the figure. Let the triangle ABC be positioned in the plane $\sum_1$ in such a way that its side AB lies in the intersection of the planes and its vertex C is in any point of $\sum_1$. If C' is the orthogonal projection of C onto the plane $\sum_2$, then the triangle ABC' is the orthogonal projection of the triangle ABC. If D is the foot of the altitude of the triangle ABC from vertex C, then $P = \frac{1}{2}|AB| \cdot |DC|$.

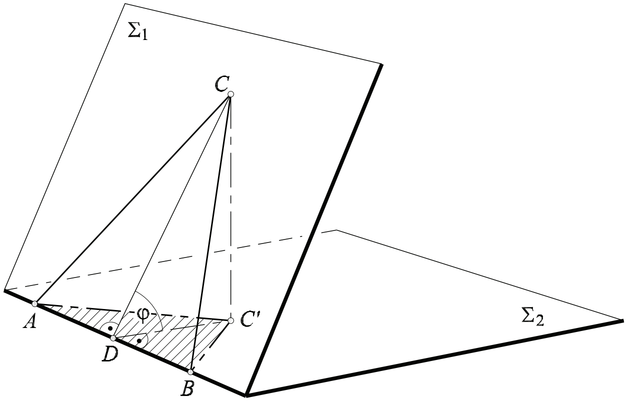

Fig. 1.4.

According to the Theorem of three perpendiculars (see Th5.), DC' is perpendicular to AB, which means that DC' is the altitude of the triangle ABC' from vertex C'. Therefore $Q = \frac{1}{2}|AB| \cdot |DC'| = \frac{1}{2}|AB| \cdot |DC| \cos\varphi$ $= P\cos\varphi$. Q.E.D.

Now we are going to explore a very important notion of the geometry of space. That notion and the theorems which are directly related thereto will be frequently used in the following text. It is the **trihedron**.

D1. *Let a, b and c be three rays in space which do not lie in the same plane and which have a common origin O. The part of space bounded by the three angles determined by those rays in pairs is called a trihedron.*

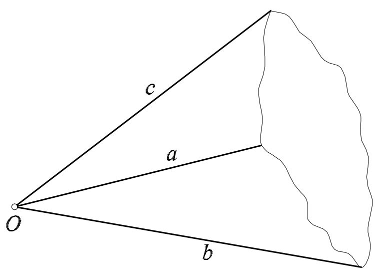

Fig. 1.5.

The point O is the vertex and the rays a, b, c are the edges of the trihedron. The parts of planes determined by pairs of edges are the faces of the trihedron. A trihedron is represented in a plane by its projection, as in Fig. 1.5. Trihedron is important in the study of the tetrahedron. Let us state an analogy between the triangle and the tetrahedron. If an angle with vertex A is intersected by a line that meets the sides of the angle in the points B and C, then the angle is divided into two parts, one bounded and one unbounded. The bounded part is the triangle ABC with vertices A, B and C.

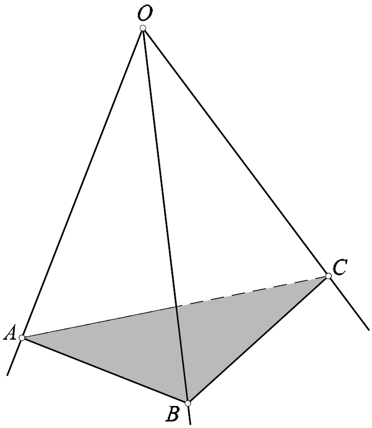

Fig. 1.6.

We could define the tetrahedron in a similar way: if a trihedron with vertex O is divided into two parts by a plane which intersects its edges in the points A, B and C, then the bounded part is a tetrahedron with vertices O, A, B and C, as shown in Fig. 1.6.

There are two types of angles defined in a trihedron:

1. *Angles determined by two edges are called* **face angles** *of a trihedron. There are three such angles in a trihedron.*
2. *Angles between two faces, or dihedral angles, determined by two faces of a trihedron. A tetrahedron obviously has three such angles.*

At this point I have to say that, unfortunately, in the Croatian mathematical terminology there is no full accordance with regard to these angles.

When we talk about the angle at a certain edge of a trihedron, we mean the angle between the two faces having that edge in common.

To each face angle we associate the angle between those two faces which share the edge that is not an edge of that face angle. We say that to each face angle we asociate the opposite angle of two faces of the trihedron.

In a trihedron with vertex O and edges a, b and c, we will denote face angles by α_1, β_1, γ_1, and angles between faces (dihedral angles) by α_1, β_1, γ_1 as it is shown in Fig. 1.7.

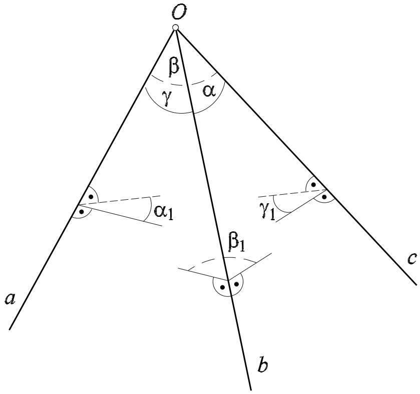

Fig. 1.7.

We can see, for instance, that the face angle α is situated opposite to the edge a, and the corresponding dihedral angle at that edge is denoted by α_1. An analogous accordance of notation holds for other edges and angles. The following theorems are valid for the face angles of a trihedron:

Th7. *Each face angle is smaller than the sum of the other two face angles and greater than the difference of the other two face angles.*

This proposition is easily proved in such a manner that one edge is orthogonally projected onto the opposite face, so that the opposite angle is divided into two parts, each of them smaller than the corresponding face angle.

Th8. *If two face angles in a trihedron are equal, then their opposite dihedral angles are equal, too.*

Hint for a proof: Project orthogonally the edge that the equal face angles have in common onto the opposite face and also project any point of that edge onto the other two edges.

Th9. *If the face angles of a trihedron satisfy the relation $\alpha < \beta < \gamma$, then for the opposite dihedral angles the relation $\alpha_1 < \beta_1 < \gamma_1$ is valid.*

This assertion, too, is easily proved by projecting orthogonally one edge of the trihedron onto the plane of its opposite face.

The previous theorems remind us of some assertions about the interior angles of a triangle. We know that the sum of such angles equals 180°.

However, the sum of face angles of a trihedron is not constant and the following theorem holds.

Th10. *The sum of the face angles of a trihedron is smaller than* 360°.

Proof. We observe the intersection of a trihedron with the vertex O and the face angles α, β and γ by any plane, in such a way that it intersects the edges of the trihedron in the points A, B and C, as shown in Fig. 1.8.

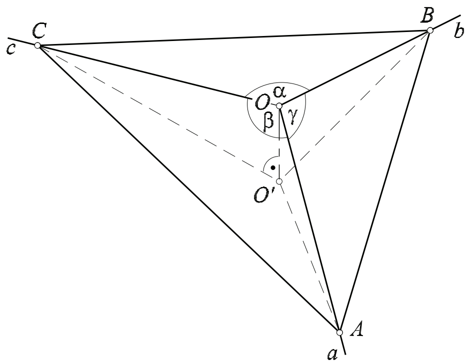

Fig. 1.8.

If O' is the orthogonal projection of the vertex O onto the plane ABC, then it obviously holds $\alpha = \sphericalangle BOC < \sphericalangle BO'C$, $\beta = \sphericalangle COA < \sphericalangle CO'A$, $\gamma = \sphericalangle AOB < \sphericalangle AO'B$. Therefrom one gets $\alpha + \beta + \gamma < \sphericalangle BO'C + \sphericalangle CO'A + \sphericalangle AO'B$, or $\alpha + \beta + \gamma < 360^\circ$. Q.E.D.

Similarly, the sum of dihedral angles of a trihedron is not constant, but the following theorem holds.

Th11. *Dihedral angles of a trihedron satisfy the relation:*

$$180^\circ < \alpha_1 + \beta_1 + \gamma_1 < 540^\circ.$$

Proof. Consider a trihedron with vertex in the point O. Let O' be any point inside the trihedron and A', B', C' projections of the point O onto the faces of the trihedron. The perpendiculars from those points onto the edges of the trihedron mutually intersect in pairs on the edges, in the points A, B and C, as represented in Fig. 1.9.

Dihedral angles of our trihedron are: $\alpha_1 = \sphericalangle C'AB'$, $\beta_1 = \sphericalangle A'BC'$, $\gamma_1 = \sphericalangle B'CA'$. If we denote the face angles of the trihedron $A'B'C'O'$ by α', β', γ' then $\alpha_1 + \alpha' = 180^\circ$, $\beta_1 + \beta' = 180^\circ$, $\gamma_1 + \gamma' = 180^\circ$.

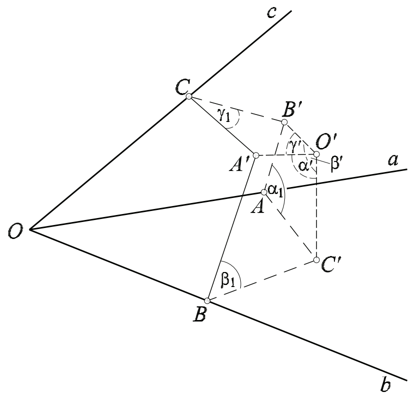

Fig. 1.9.

By adding up one gets $(\alpha_1 + \beta_1 + \gamma_1) + (\alpha' + \beta' + \gamma') = 540^\circ$. An application of the theorem Th10. to the tetrahedron $A'B'C'O'$ directly yields the assertion of the theorem. Q.E.D.

The following theorem shows how to calculate the dihedral angles of a trihedron if the values of face angles are given.

Th12. *If α, β and γ are face angles, and α_1, β_1 and γ_1 are dihedral angles of a trihedron, then*

$$\cos\alpha_1 = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}, \qquad \cos\beta_1 = \frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha},$$
$$\cos\gamma_1 = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

Proof. On the edges of a trihedron with vertex in the point O determine the points A, B and C such that the vectors $\vec{e}_1 = \overrightarrow{OA}$, $\vec{e}_2 = \overrightarrow{OB}$, $\vec{e}_3 = \overrightarrow{OC}$ be unit vectors. Let D and E be the feet of the perpendiculars from the points B and C to the line OA. Then, by definition of a dihedral angle, $\alpha_1 = \measuredangle(\overrightarrow{DB}, \overrightarrow{EC})$ (Fig. 1.10.).

It holds $\overrightarrow{DB} = \overrightarrow{DO} + \vec{e}_2$, $\overrightarrow{EC} = \overrightarrow{EO} + \vec{e}_3$. Since $\overrightarrow{DO} \cdot \vec{e}_3 = |DO| \cdot 1 \cdot \cos(-\beta) = -\cos\gamma\cos\beta$, $\overrightarrow{EO} \cdot \vec{e}_2 = -\cos\beta\cos\gamma$, it follows $\overrightarrow{DB} \cdot \overrightarrow{EC} = \cos\gamma\cos\beta - \cos\beta\cos\gamma - \cos\beta\cos\gamma + \cos\alpha$. Finally, owing to $\overrightarrow{DB} \cdot \overrightarrow{EC} = \sin\gamma\sin\beta\cos\alpha_1$, we obtain $\cos\alpha_1 =$

$\dfrac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$. The other two formulas of the theorem can be proved in an analogous manner.

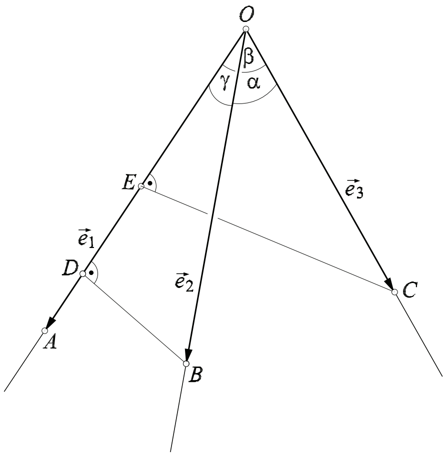

Fig. 1.10.

Q.E.D.

Let us state the formula for the volume of a tetrahedron determined by vectors of its edges with the origin in a common vertex:

$$V = \frac{1}{6}\left|\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\,\right| = \frac{1}{6}\left\|\begin{matrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{matrix}\right\|. \tag{F2}$$