Chapter 1

Preliminaries

In this chapter, we review the basic concepts of a Hilbert space and (bounded linear) operators on a Hilbert space, which will recur throughout the book.

1.1 Hilbert space and operators

Definition 1.1 *A complex vector space H is called an inner product space if to each pairs of vectors x and y in H is associated a complex number $\langle x,y\rangle$, called the inner product of x and y, such that the following rules hold:*

(i) *For $x,y \in H$, $\langle x,y\rangle = \overline{\langle y,x\rangle}$, where the bar denotes complex conjugation.*

(ii) *If x,y and $z \in H$ and $\alpha,\beta \in \mathbb{C}$, then $\langle \alpha x+\beta y,z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$.*

(iii) *$\langle x,x\rangle \geq 0$ for all $x \in H$ and equal to zero if and only if x is the zero vector.*

Theorem 1.1 (SCHWARZ INEQUALITY) *Let H be an inner product space. If x and $y \in H$, then*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle \tag{1.1}$$

and the equality holds if and only if x and y are linearly dependent.

Proof. If $y = 0$, then the inequality (1.1) holds. Suppose that $y \neq 0$ and put

$$e = \frac{1}{\sqrt{\langle y, y\rangle}} y.$$

Then we have

$$\begin{aligned} 0 &\leq \langle x - \langle x, e\rangle e, x - \langle x, e\rangle e\rangle \\ &= \langle x, x\rangle - \overline{\langle x, e\rangle}\langle x, e\rangle - \langle x, e\rangle\langle e, x\rangle + |\langle x, e\rangle|^2 \langle e, e\rangle \\ &= \langle x, x\rangle - 2|\langle x, e\rangle|^2 + |\langle x, e\rangle|^2 \\ &= \langle x, x\rangle - |\langle x, e\rangle|^2 \end{aligned}$$

and hence $|\langle x, e\rangle|^2 \leq \langle x, x\rangle$. Therefore it follows that $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$.

If the equality holds in the inequality above, then we have $x - \langle x, e\rangle e = 0$, and so x and y are linearly dependent. Conversely, if x and y are linearly dependent, that is, there exists a constant $\alpha \in \mathbb{C}$ such that $x = \alpha y \neq 0$, then it follows that

$$|\langle x, y\rangle|^2 = |\langle \alpha y, y\rangle|^2 = |\alpha|^2 |\langle y, y\rangle|^2 = \langle \alpha y, \alpha y\rangle\langle y, y\rangle = \langle x, x\rangle\langle y, y\rangle.$$

We can prove it in the case of $y = \alpha x$ in the same way. □

Let H be an inner product space. Put

$$\|x\| = \sqrt{\langle x, x\rangle} \qquad \text{for all } x \in H.$$

Then it follows that $\|\cdot\|$ is a norm on H:

(i) Positivity: $\|x\| \geq 0$ and $x = 0$ if and only if $\|x\| = 0$.

(ii) Homogeneity: $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{C}$.

(iii) Triangular inequality: $\|x + y\| \leq \|x\| + \|y\|$.

In fact, positivity and homogeneity are obvious by Definition 1.1. Triangular inequality follows from

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\mathrm{Re}\langle x, y\rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \end{aligned}$$

by Schwarz's inequality (Theorem 1.1). Therefore, $\|x\|$ is a norm on H.

Definition 1.2 *If an inner product space H is complete with respect to the norm derived from the inner product, then H is said to be a Hilbert space.*

Some examples of Hilbert spaces will now be given.

Example 1.1 *The space $\mathbb{C}^n$ of all n-tuples of complex numbers with the inner product between $x = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $y = (\beta_1, \beta_2, \cdots, \beta_n)$ given by*

$$\langle x, y \rangle = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

is a Hilbert space.

Example 1.2 *The space l_2 of all sequences of complex numbers $(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ with*

$$\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$$

and the inner product between $x = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ and $y = (\beta_1, \beta_2, \cdots, \beta_n, \cdots)$ given by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}$$

is a Hilbert space.

A linear operator A on a Hilbert space H is said to be bounded if there exists $c > 0$ such that $\|Ax\| \leq c\|x\|$ for all $x \in H$. Let us define $\|A\|$ by

$$\|A\| = \inf\{c > 0 : \|Ax\| \leq c\|x\| \quad \text{for all } x \in H.\}$$

Then $\|A\|$ is said to be the operator norm of A. By definition,

$$\|Ax\| \leq \|A\|\|x\| \qquad \text{for all } x \in H.$$

In fact, for each $x \neq 0$, $\|Ax\| \leq c\|x\|$ implies $\frac{\|Ax\|}{\|x\|} \leq c$. Taking the inf of c, we have $\frac{\|Ax\|}{\|x\|} \leq \|A\|$.

We begin by adopting the word "operator" to mean a bounded linear operator.

$B(H)$ will now denote the algebra of all bounded linear operators on a Hilbert space $H \neq \{0\}$ and I_H stands for the identity operator.

The following lemma shows some characterizations of the operator norm.

Lemma 1.1 *For any operator $A \in B(H)$, the following formulae hold:*

$$\begin{aligned} \|A\| &= \sup\{\|Ax\| \ : \ \|x\| = 1, \ x \in H\} \\ &= \sup\left\{\frac{\|Ax\|}{\|x\|} \ : \ x \neq 0, \ x \in H\right\} \\ &= \sup\{|\langle Ax, y\rangle| \ : \ \|x\| = \|y\| = 1, \ x, y \in H\} \end{aligned}$$

Proof. Put

$$\gamma_1 = \sup\{\|Ax\| : \|x\| = 1\} \quad \text{and} \quad \gamma_2 = \sup\left\{\frac{\|Ax\|}{\|x\|} : x \neq 0\right\}.$$

For $\|x\| = 1$, we have $\|Ax\| \le \|A\|\|x\| = \|A\|$ and hence $\gamma_1 \le \|A\|$. For $x \ne 0$, we have

$$\frac{\|Ax\|}{\|x\|} = \|A\frac{x}{\|x\|}\| \le \gamma_1$$

and hence $\gamma_2 \le \gamma_1$. For an arbitrary $\varepsilon > 0$, there exists a nonzero vector $x \in H$ such that $(\|A\| - \varepsilon)\|x\| < \|Ax\|$ and hence

$$\|A\| - \varepsilon < \frac{\|Ax\|}{\|x\|} \le \gamma_2.$$

This fact implies $\|A\| \le \gamma_2$. Therefore we have $\|A\| = \gamma_1 = \gamma_2$.

Put

$$\gamma_3 = \sup\{|\langle Ax, y\rangle| : \|x\| = 1, \|y\| = 1\}.$$

Since $|\langle Ax, y\rangle| \le \|Ax\|\|y\| = \|Ax\| \le \gamma_1$ for $\|x\| = \|y\| = 1$, we have $\gamma_3 \le \gamma_1$. Conversely, for $Ax \ne 0$, we have

$$\|Ax\| = |\langle Ax, \frac{Ax}{\|Ax\|}\rangle| \le \gamma_3$$

and hence $\gamma_1 \le \gamma_3$. Therefore the proof is complete. □

Theorem 1.2 *The following properties hold for $A, B \in B(H)$:*

(i) *If $A \ne O$, then $\|A\| > 0$,*

(ii) $\|\alpha A\| = |\alpha|\|A\|$ *for all* $\alpha \in \mathbb{C}$,

(iii) $\|A + B\| \le \|A\| + \|B\|$,

(iv) $\|AB\| \le \|A\|\|B\|$.

Proof.
(i) If $A \ne O$, then there exists a nonzero vector $x \in H$ such that $Ax \ne 0$. Hence $0 < \|Ax\| \le \|A\|\|x\|$, therefore $\|A\| > 0$.
(ii) If $\alpha = 0$, then $\|\alpha A\| = \|O\| = 0 = |\alpha|\|A\|$. If $\alpha \ne 0$, then

$$\begin{aligned}\|\alpha A\| &= \sup\{\|(\alpha A)x\| : \|x\| = 1\}\\ &= \sup\{|\alpha|\|Ax\| : \|x\| = 1\}\\ &= |\alpha|\sup\{\|Ax\| : \|x\| = 1\} = |\alpha|\|A\|.\end{aligned}$$

(iii) If $\|x\| = 1$, then $\|(A+B)x\| = \|Ax + Bx\| \le \|Ax\| + \|Bx\| \le \|A\| + \|B\|$, therefore we have

$$\|A + B\| = \sup\{\|(A+B)x\| : \|x\| = 1\} \le \|A\| + \|B\|.$$

(iv) If $\|x\| = 1$, then $\|(AB)x\| = \|A(Bx)\| \le \|A\|\|Bx\| \le \|A\|\|B\|$, therefore we have

$$\|AB\| = \sup\{\|(AB)x\| : \|x\| = 1\} \le \|A\|\|B\|.$$

□

Theorem 1.3 (RIESZ REPRESENTATION THEOREM) *For each bounded linear functional f from H to $\mathbb{C}$, there exists a unique $y \in H$ such that*

$$f(x) = \langle x, y \rangle \qquad \text{for all } x \in H.$$

Moreover, $\|f\| = \|y\|$.

Proof. Define $\mathscr{M} = \{x \in H : f(x) = 0\}$. Then $\mathscr{M}$ is closed. If $\mathscr{M} = H$, then $f = 0$ and we can choose $y = 0$. If $\mathscr{M} \neq H$, then $\mathscr{M}^{\perp} \neq \{0\}$. For $x_0 \in \mathscr{M}^{\perp} \backslash \{0\}$, we have $f(x_0) \neq 0$. Since

$$f(x - \frac{f(x)}{f(x_0)} x_0) = f(x) - \frac{f(x)}{f(x_0)} f(x_0) = 0 \qquad \text{for all } x \in H,$$

it follows that $x - \frac{f(x)}{f(x_0)} x_0 \in \mathscr{M}$. Hence we have

$$\langle x - \frac{f(x)}{f(x_0)} x_0, x_0 \rangle = 0$$

and $\langle x, x_0 \rangle = \frac{f(x)}{f(x_0)} \|x_0\|^2$. If we put $y = \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0$, then we have $f(x) = \langle x, y \rangle$ for all $x \in H$.

For the uniqueness, suppose that $f(x) = \langle x, y \rangle = \langle x, z \rangle$ for all $x \in H$. In this case, $\langle x, y - z \rangle = 0$ for all $x \in H$ implies $y - z = 0$.

Finally,

$$|f(x)| = |\langle x, y \rangle| \leq \|x\| \|y\|$$

implies $\|f\| \leq \|y\|$. Conversely,

$$\|y\|^2 = |\langle y, y \rangle| = |f(y)| \leq \|f\| \|y\|$$

implies $\|y\| \leq \|f\|$. Therefore, we have $\|f\| = \|y\|$. □

For a fixed $A \in B(H)$, a functional on H defined by

$$x \mapsto \langle Ax, y \rangle \in \mathbb{C}$$

is bounded linear on H. By the Riesz representation theorem, there exists a unique $y^* \in H$ such that

$$\langle Ax, y \rangle = \langle x, y^* \rangle \qquad \text{for all } x \in H.$$

We now define

$$A^* : y \mapsto y^*,$$

the mapping A^* being called the adjoint of A. In summary,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \text{for all } x, y \in H.$$

Theorem 1.4 *The adjoint operation is closed in $B(H)$ and moreover*

(i) $\|A^*\| = \|A\|$,

(ii) $\|A^* A\| = \|A\|^2$.

Proof.
(i): For $y_1, y_2 \in H$ and $\alpha_1, \alpha_2 \in \mathbb{C}$,

$$\begin{aligned}\langle Ax, \alpha_1 y_1 + \alpha_2 y_2\rangle &= \overline{\alpha_1}\langle Ax, y_1\rangle + \overline{\alpha_2}\langle Ax, y_2\rangle \\ &= \overline{\alpha_1}\langle x, A^* y_1\rangle + \overline{\alpha_2}\langle x, A^* y_2\rangle \\ &= \langle x, \alpha_1 A^* y_1 + \alpha_2 A^* y_2\rangle \qquad \text{for all } x \in H.\end{aligned}$$

This implies $A^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^* y_1 + \alpha_2 A^* y_2$ and A^* is linear. Next,

$$\begin{aligned}\|A^* y\| &= \sup\{|\langle x, A^* y\rangle| : \ \|x\| = 1\} \\ &= \sup\{|\langle Ax, y\rangle| : \ \|x\| = 1\} \\ &\leq \sup\{\|Ax\|\|y\| : \ \|x\| = 1\} = \|A\|\|y\|,\end{aligned}$$

hence A^* is bounded and $\|A^*\| \leq \|A\|$. Therefore, the adjoint operation is closed in $B(H)$. Since $(A^*)^* = A$, we have

$$\|A\| = \|(A^*)^*\| \leq \|A^*\|$$

and hence $\|A^*\| = \|A\|$.
(ii): Since $\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^* Ax, x\rangle \leq \|A^* A\|\|x\|^2$ for every $x \in H$, we have $\|A\|^2 \leq \|A^* A\|$.

On the other hand, (i) gives $\|A^* A\| \leq \|A^*\|\|A\| = \|A\|^2$. Hence the equality

$$\|A^* A\| = \|A\|^2$$

holds for every $A \in B(H)$. □

1.2 Self-adjoint operators

We present relevant classes of operators:

Definition 1.3 *An operator $A \in B(H)$ is said to be*

(i) *self-adjoint or Hermitian if $A = A^*$,*

(ii) *positive if $\langle Ax, x\rangle \geq 0$ for x in H,*

(iii) *unitary if $A^* A = AA^* = I_H$,*

(iv) *isometry if $A^* A = I_H$,*

(v) *projection if $A = A^* = A^2$.*

The following theorem gives characterizations of self-adjoint operators.

Theorem 1.5 *If $A \in B(H)$, the following three statements are mutually equivalent.*

(i) *A is self-adjoint.*

(ii) *$\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$.*

(iii) *$\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in H$.*

Proof.
(i) $\Longleftrightarrow$ (ii): If A is self-adjoint, then $\langle Ax, y\rangle = \langle x, A^*y\rangle = \langle x, Ay\rangle$. Conversely suppose that (ii) holds. Since $\langle x, A^*y\rangle = \langle x, Ay\rangle$ for all $x, y \in H$, we have $A^*y = Ay$, so that $A = A^*$.
(ii) $\Longleftrightarrow$ (iii): If we put $y = x$ in (ii), then

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle},$$

so $\langle Ax, x\rangle$ is real. Thus (ii) implies (iii). Finally, suppose that (iii) holds. For each x and $y \in H$, if we put $w = x + y$, then $\langle Aw, w\rangle$ is real, or $\langle Aw, w\rangle = \langle w, Aw\rangle$. Expanding $\langle A(x+y), x+y\rangle = \langle x+y, A(x+y)\rangle$, we have

$$\langle Ax, y\rangle + \langle Ay, x\rangle = \langle x, Ay\rangle + \langle y, Ax\rangle$$

and $\mathrm{Im}\langle Ax, y\rangle = \mathrm{Im}\langle x, Ay\rangle$. Replacing x by ix, we have $\mathrm{Re}\langle Ax, y\rangle = \mathrm{Re}\langle x, Ay\rangle$. Therefore it follows that $\langle Ax, y\rangle = \langle x, Ay\rangle$. Thus (iii) implies (ii). □

The spectrum of an operator A is the set

$$\mathsf{Sp}(A) = \{\lambda \in \mathbb{C} : A - \lambda I_H \text{ is not invertible in } B(H)\}.$$

The spectrum $\mathsf{Sp}(A)$ is nonempty and compact. An operator A on a Hilbert space H is bounded below if there exists $\varepsilon > 0$ such that $\|Ax\| \geq \varepsilon\|x\|$ for every $x \in H$. As a useful criterion for the invertibility of an operator, it is well known that A is invertible if and only if both A and A^* are bounded below.

The spectral radius $r(A)$ of an operator A is defined by

$$r(A) = \sup\{|\alpha| : \alpha \in \mathsf{Sp}(A)\}.$$

Then we have the following relation between the operator norm and the spectral radius.

Theorem 1.6 *For an operator A, the spectral radius is not greater than the operator norm:*

$$r(A) \leq \|A\|.$$

Proof. If $|\alpha| > \|A\|$, then $I_H - \alpha^{-1}A$ is invertible and hence $A - \alpha I_H$ is so. Therefore we have $\alpha \notin \mathsf{Sp}(A)$ and this implies $r(A) \leq \|A\|$. □

Let A be a self-adjoint operator on a Hilbert space H. We define

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle. \tag{1.2}$$

Theorem 1.7 *For a self-adjoint operator A, $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.*

Proof. If $\lambda = \alpha + i\beta$ with α, β real and $\beta \neq 0$, then we must show that $A - \lambda I_H$ is invertible. Put $B = \frac{1}{\beta}(A - \alpha I_H)$. Since B is self-adjoint and $B - iI_H = \frac{1}{\beta}(A - \lambda I_H)$, it follows that $A - \lambda I_H$ is invertible if and only if $B - iI_H$ is invertible. For every $x \in H$, we have

$$\begin{aligned} \|(B \pm iI_H)x\|^2 &= \|Bx\|^2 - i\langle x, Bx\rangle + i\langle Bx, x\rangle + \|x\|^2 \\ &= \|Bx\|^2 + \|x\|^2 \geq \|x\|^2, \end{aligned}$$

so $B - iI_H$ and $(B - iI_H)^*$ are bounded below. Therefore $B - iI_H$ is invertible, and hence the spectrum of a self-adjoint operator is real.

Next, to prove $\mathsf{Sp}(A) \subset [m_A, M_A]$, it is enough to show that $\lambda > M_A$ implies $\lambda \notin \mathsf{Sp}(A)$. If $\lambda > M_A$ and $\varepsilon = \lambda - M_A > 0$, then

$$\begin{aligned} \langle(\lambda I_H - A)x, x\rangle &= \lambda\langle x, x\rangle - \langle Ax, x\rangle \geq \lambda\langle x, x\rangle - M_A\langle x, x\rangle \\ &= \varepsilon\langle x, x\rangle \geq 0 \qquad \text{by the definition of } M_A. \end{aligned}$$

Hence it follows that $\|(A - \lambda I_H)x\| \geq \varepsilon\|x\|$ for every $x \in H$, so, $A - \lambda I_H$ is bounded below. Since $A - \lambda I_H$ is self-adjoint, it follows that $A - \lambda I_H$ is invertible and $\lambda \notin \mathsf{Sp}(A)$. □

Definition 1.4 *Let A and B be self-adjoint operators on H. We write $A \geq B$ if $A - B$ is positive, i.e. $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for every $x \in H$. In particular, we write $A \geq 0$ if A is positive, $A > 0$ if A is positive and invertible.*

Now, we review the continuous functional calculus. A rudimentary functional calculus for an operator A can be defined as follows: For a polynomial $p(t) = \sum_{j=0}^{k} \alpha_j t^j$, define

$$p(A) = \alpha_0 I_H + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_k A^k.$$

The mapping $p \to p(A)$ is a homomorphism from the algebra of polynomials to the algebra of operators. The extension of this mapping to larger algebras of functions is really significant in operator theory.

Let A be a self-adjoint operator on a Hilbert space H. Then the Gelfand mapping establishes a $*$-isometrically isomorphism Φ between C*-algebra $C(\mathsf{Sp}(A))$ of all continuous functions on $\mathsf{Sp}(A)$ and C*-algebra $C^*(A)$ generated by A and the identity operator I_H on H as follows: For $f, g \in C(\mathsf{Sp}(A))$ and $\alpha, \beta \in \mathbb{C}$

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$,

(iii) $\|\Phi(f)\| = \|f\| \left(:= \sup_{t \in \mathsf{Sp}(A)} |f(t)| \right)$,

(iv) $\Phi(f_0) = I_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$.

With this notation, we define

$$f(A) = \Phi(f)$$

for all $f \in C(\mathsf{Sp}(A))$ and we call it the continuous functional calculus for a self-adjoint operator A. It is an extension of $p(A)$ for a polynomial p. The continuous functional calculus is applicable.

Theorem 1.8 *Let A be a self-adjoint operator on H.*

(i) $f \in C(\mathsf{Sp}(A))$ *and* $f \geq 0$ *implies* $f(A) \geq 0$.

(ii) $f, g \in C(\mathsf{Sp}(A))$ *and* $f \geq g$ *implies* $f(A) \geq g(A)$.

(iii) $A \geq 0$ *and* $f_{1/2}(t) = \sqrt{t}$ *implies* $f_{1/2}(A) = A^{1/2}$.

(iv) $f_s(t) = |t|$ *implies* $f_s(A) = |A|$.

Proof.
(i) Since $f \geq 0$, we can choose $g = \sqrt{f} \in C(\mathsf{Sp}(A))$ and $f = g^2 = \overline{g}g$. Hence we have $f(A) = g(A)^* g(A) \geq 0$.
(ii) follows from (i).
(iii) Since $A \geq 0$, it follows from Theorem 1.7 that $f_{1/2}(t) = \sqrt{t} \in C(\mathsf{Sp}(A))$. Also, $f_1 = f_{1/2}^2$ implies $A = f_1(A) = f_{1/2}(A)^2$. By (i), we have $f_{1/2}(A) \geq 0$ and hence $f_{1/2}(A) = A^{1/2}$.
(iv) $f_s^2 = f_1^2$ implies $f_s(A)^2 = A^2 = |A|^2$. Since $f_s(A) \geq 0$, we have $f_s(A) = |A|$. □

We remark that the absolute value of an operator A is defined by $|A| = (A^*A)^{1/2}$.

Theorem 1.9 *An operator A is positive if and only if there is an operator B such that $A = B^*B$.*

Proof. If A is positive, take $B = \sqrt{A}$. If $A = B^*B$, then $\langle Ax, x\rangle = \langle B^*Bx, x\rangle = \|Bx\|^2 \geq 0$ for every $x \in H$. This yields that A is positive. □

Theorem 1.10 (GENERALIZED SCHWARZ'S INEQUALITY) *If A is positive, then*

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle \langle Ay, y\rangle$$

for every $x, y \in H$.

Proof. It follows from Theorem 1.1 that

$$|\langle Ax, y\rangle|^2 = |\langle A^{1/2}x, A^{1/2}y\rangle|^2 \leq \|A^{1/2}x\|^2 \|A^{1/2}y\|^2 = \langle Ax, x\rangle \langle Ay, y\rangle.$$

□

Theorem 1.11 *Let A be a self-adjoint operator on H. Then*

(i) $m_A I_H \leq A \leq M_A I_H$,

(ii) $\|A\| = \max\{|m_A|, |M_A|\} = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}$,
where m_A and M_A are defined by (1.2).

Proof. The assertion (i) is clear by definition of m_A and M_A.
Next, put $K = \max\{|m_A|, |M_A|\}$. It is easily checked that

$$K = \sup\{|\langle Ax, x\rangle| : \ \|x\| = 1\} \leq \|A\|.$$

By (i), we have

$$-K\|x\|^2 \leq m\|x\|^2 \leq \langle Ax, x\rangle \leq M\|x\|^2 \leq K\|x\|^2.$$

For each $x, y \in H$, since

$$|\langle A(x+y), x+y\rangle| \leq K\|x+y\|^2 \quad \text{and} \quad |\langle A(x-y), x-y\rangle| \leq K\|x-y\|^2,$$

it follows that

$$|\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle| \leq K(\|x+y\|^2 + \|x-y\|^2).$$

By the parallelogram identity, we have

$$4|\mathrm{Re}\langle Ax, y\rangle| \leq 2K(\|x\|^2 + \|y\|^2). \tag{1.3}$$

Put $y = \frac{\|x\|}{\|Ax\|}Ax$ for $Ax \neq 0$. Then $\|x\| = \|y\|$ and $\mathrm{Re}\langle Ax, y\rangle = \|x\|\|Ax\|$. Therefore, by (1.3) we have

$$\|Ax\| \leq K\|x\|. \tag{1.4}$$

If $Ax = 0$, then (1.4) holds automatically. Hence we have $\|A\| \leq K$. Therefore we have $\|A\| = K$. □

Corollary 1.1 *If A is a self-adjoint operator, then $r(A) = \|A\|$ and $\|A^n\| = \|A\|^n$ for $n \in \mathbb{N}$.*

Proof. By Theorem 1.11, it follows that $r(A) = \|A\|$. By the spectral mapping theorem, we have $p(\mathsf{Sp}(A)) = \mathsf{Sp}(p(A))$ for polynomial p. Therefore, we have $\|A\|^n = r(A)^n = r(A^n) = \|A^n\|$. □

1.3 Spectral decomposition theorem

We shall introduce the spectral decomposition theorem for self-adjoint, bounded linear operators on a Hilbert space H. To show it, we need the following notation and lemma.

Definition 1.5 *If A is an operator on a Hilbert space H, then the kernel of A, denoted by* $\ker A$*, is the closed subspace $\{x \in H : Ax = 0\}$, and the range of A, denoted by* $\mathrm{ran}\, A$*, is the subspace $\{Ax : x \in H\}$.*

Lemma 1.2 *If A is an operator on a Hilbert space H, then*

$$\ker A = (\operatorname{ran} A^*)^\perp \quad \textit{and} \quad \ker A^* = (\operatorname{ran} A)^\perp.$$

Proof. If $x \in \ker A$, then $\langle A^*y, x\rangle = \langle y, Ax\rangle = 0$ for all $y \in H$, and hence x is orthogonal to $\operatorname{ran} A^*$. Conversely, if x is orthogonal to $\operatorname{ran} A^*$, then $\langle Ax, y\rangle = \langle x, A^*y\rangle = 0$ for all $y \in H$, which implies $Ax = 0$. Therefore, $x \in \ker A$ and hence $\ker A = (\operatorname{ran} A^*)^\perp$. We have the second relation by replacing A by A^*. □

Definition 1.6 *A family of projections $\{e(\lambda) : \lambda \in \mathbb{R}\}$ is said to be a resolution of the identity if the following properties hold:*

(i) $\lambda < \lambda' \implies e(\lambda) \le e(\lambda')$,

(ii) $e(-\infty) = O$ *and* $e(\infty) = I_H$,

(iii) $e(\lambda + 0) = e(\lambda) \qquad (-\infty < \lambda < \infty)$,
where $e(\lambda + 0) = \text{s}-\lim_{\mu \to \lambda + 0} e(\mu)$.

Theorem 1.12 *Let A be a self-adjoint operator on a Hilbert space H and $m = m_A, M = M_A$ as defined by* (1.2)*. Then there exists a resolution of the identity $\{e(\lambda) : \lambda \in \mathbb{R}\}$ such that*

$$A = \int_{m-0}^{M} \lambda \, \mathrm{d}e(\lambda), \quad e(m-0) = 0 \quad \textit{and} \quad e(M) = I_H.$$

In particular,

$$\langle Ax, x\rangle = \int_{m-0}^{M} \lambda \, \mathrm{d}\langle e(\lambda)x, x\rangle \qquad \textit{for every } x \in H. \tag{1.5}$$

Proof. We prove only (1.5). Put $e(\lambda) = \operatorname{proj}(\ker((A - \lambda I_H)^+))$ for $\lambda \in \mathbb{R}$, where $A^+ = (|A| + A)/2$. Then it follows that $\{e(\lambda) : \lambda \in \mathbb{R}\}$ is a resolution of the identity and $e(m-0) = 0$, $e(M) = I_H$:
(i) Let $\lambda < \lambda'$. Since $A - \lambda I_H \ge A - \lambda' I_H$, we have $(A - \lambda I_H)^+ \ge (A - \lambda' I_H)^+ \ge 0$. If $(A - \lambda I_H)^+ x = 0$, then

$$0 = \langle (A - \lambda I_H)^+ x, x\rangle \ge \langle (A - \lambda' I_H)^+ x, x\rangle \ge 0$$

and hence $(A - \lambda' I_H)x = 0$. Therefore, we have $\ker((A - \lambda I_H)^+) \subset \ker((A - \lambda' I_H)^+)$ and this implies $e(\lambda) \le e(\lambda')$.
(ii) If $x \in \operatorname{ran}(e(\lambda)) = \ker((A - \lambda I_H)^+)$, then $(A - \lambda I_H)^+ x = 0$ implies $(A - \lambda I_H)x = -(A - \lambda I_H)^- x$ and hence

$$\langle (A - \lambda I_H)x, x\rangle = -\langle (A - \lambda I_H)^- x, x\rangle \le 0.$$

Therefore we have $\langle Ax, x\rangle \le \lambda \|x\|^2$.
(iii) If $x \in \operatorname{ran}(I_H - e(\lambda)) = (\ker((A - \lambda I_H)^+)^\perp$, then $(A - \lambda I_H)^- x \in \ker((A - \lambda I_H)^+)$ because $(A - \lambda I_H)^+ (A - \lambda I_H)^- = 0$. Hence $\langle (A - \lambda I_H)^- x, x\rangle = 0$ and $\langle (A - \lambda I_H)x, x\rangle =$

$\langle (A-\lambda I_H)^+x,x\rangle \geq 0$. Therefore we have $\langle Ax,x\rangle \geq \lambda\|x\|^2$. If the equality holds, then $\langle (A-\lambda I_H)^+x,x\rangle = 0$ and hence $(A-\lambda I_H)x = 0$. Therefore we have $x \in \ker((A-\lambda I_H)^+)$ and hence $x = 0$. Summing up, $x \in \operatorname{ran}(I_H - e(\lambda))$, $x \neq 0$ implies $\langle Ax,x\rangle > \lambda\|x\|^2$.

(iv) If $\lambda < m$ and $x \in \operatorname{ran}(e(\lambda))$, then it follows from (ii) that $m\|x\|^2 \leq \langle Ax,x\rangle \leq \lambda\|x\|^2$ and hence $x = 0$. Therefore we have $e(\lambda) = O$, so that $e(m-0) = O$.

(v) If $\lambda \geq M$ and $x \in \operatorname{ran}(I_H - e(\lambda))$, then it follows from (iii) that $\lambda\|x\|^2 \leq \langle Ax,x\rangle \leq M\|x\|^2$ and hence $x = 0$. Therefore we have $I_H - e(\lambda) = O$, so that $e(\lambda) = I_H$. In particular, we have $e(M) = I_H$.

(vi) If $\lambda < m$ or $\lambda \geq M$, then it follows from (iv), (v) that $e(\lambda) = e(\lambda - 0)$. Suppose that $m \leq \lambda < M$. Put $P = e(\lambda - 0) - e(\lambda)$. For $\lambda < \lambda' < M$, we have $\operatorname{ran}(P) \subset \operatorname{ran}(e(\lambda') - e(\lambda))') - e(\lambda)) = \operatorname{ran}(e(\lambda')) \cap \operatorname{ran}(I_H - e(\lambda))$. Hence $x \in \operatorname{ran}(P)$ and $x \neq 0$ implies $\lambda\|x\|^2 < \langle Ax,x\rangle \leq \lambda'\|x\|^2$ by (ii) and (iii). As $\lambda' \to \lambda + 0$, we get $\lambda\|x\|^2 < \lambda\|x\|^2$, which is a contradiction. Therefore we have $\operatorname{ran}(P) = \{0\}$, so that $P = e(\lambda+0) - e(\lambda) = O$.

For all $\varepsilon > 0$, we choose $\delta > 0$ such that

$$\Delta : \alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \beta, \quad \xi_k \in [\lambda_{k-1}, \lambda_k] \quad k = 1, \cdots, n,$$

and

$$|\Delta| = \max\{\lambda_k - \lambda_{k-1} : k = 1, \cdots, n\} < \delta.$$

Since A commutes with $e(\lambda)$ for each $\lambda \in \mathbb{R}$, it follows that

$$A = \sum_{k=1}^{n} A\left(e(\lambda_k) - e(\lambda_{k-1})\right).$$

For every $x \in H$, we have

$$\begin{aligned}
&\left|\langle Ax,x\rangle - \sum_{k=1}^{n} \xi_k \langle (e(\lambda_k) - e(\lambda_{k-1}))x,x\rangle\right| \\
&= \left|\sum_{k=1}^{n} \langle A(e(\lambda_k) - e(\lambda_{k-1}))x,x\rangle - \sum_{k=1}^{n} \xi_k \langle (e(\lambda_k) - e(\lambda_{k-1}))x,x\rangle\right| \\
&\leq \sum_{k=1}^{n} |\langle (A - \xi_k I)(e(\lambda_k) - e(\lambda_{k-1}))x, (e(\lambda_k) - e(\lambda_{k-1}))x\rangle| \\
&\leq \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\|(e(\lambda_k) - e(\lambda_{k-1}))x\|^2 \\
&\leq |\Delta|\|x\|^2 \leq \varepsilon.
\end{aligned}$$

Hence we have the desired result $\langle Ax,x\rangle = \int_{m-0}^{M} \lambda \, \mathrm{d}\langle e(\lambda)x,x\rangle$. □

Definition 1.7 *Let A be a self-adjoint operator on a Hilbert space H and $m = m_A, M = M_A$ as defined by* (1.2)*. For a real valued continuous function $f(\lambda)$ on $[m,M]$, a self-adjoint operator $f(A)$ is defined by*

$$f(A) = \int_{m-0}^{M} f(\lambda)\,\mathrm{d}e(\lambda).$$

In particular,

$$A^r = \int_{m-0}^{M} \lambda^r \, \mathrm{d}e(\lambda) \quad \textit{for all } r > 0 \textit{ and} \quad A^{\frac{1}{2}} = \int_{m-0}^{M} \lambda^{\frac{1}{2}} \, \mathrm{d}e(\lambda).$$

In the last part of this chapter, we present the polar decomposition for an operator.

Every complex number can be written as the product of a nonnegative number and a number of modulus one:

$$z = |z|\mathrm{e}^{i\theta} \quad \text{for a complex number } z.$$

We shall attempt a similar argument for operators on an infinite dimensional Hilbert space. Before considering this result, we need to introduce the notion of a partial isometry.

Definition 1.8 *An operator V on a Hilbert space H is a partial isometry if $\|Vx\| = \|x\|$ for $x \in (\ker V)^{\perp}$, which is called the initial space of V.*

We consider a useful characterization of partial isometries:

Lemma 1.3 *Let V be an operator on a Hilbert space H. The following are equivalent:*

(i) *V is a partial isometry.*

(ii) *V^* is a partial isometry.*

(iii) *V^*V is a projection.*

(iv) *VV^* is a projection.*

Moreover, if V is a partial isometry, then VV^* is the projection onto the range of V, while V^*V is the projection onto the initial space.

Proof. Suppose that V is a partial isometry. Since

$$\langle (I - V^*V)x, x\rangle = \langle x, x\rangle - \langle V^*Vx, x\rangle = \|x\|^2 - \|Vx\|^2 \quad \text{for } x \in H,$$

it follows that $I - V^*V$ is a positive operator. Now if x is orthogonal to $\ker V$, then $\|Vx\| = \|x\|$ which implies that $\langle (I - V^*V)x, x\rangle = 0$. Since $\|(I - V^*V)^{1/2}x\|^2 = \langle (I - V^*V)x, x) = 0$, we have $(I - V^*V)x = 0$ or $V^*Vx = x$. Therefore, V^*V is the projection onto the initial space of V.

Conversely, if V^*V is a projection and x is orthogonal to $\ker V^*V$, then $V^*Vx = x$. Therefore,

$$\|Vx\|^2 = \langle V^*Vx, x\rangle = \langle x, x\rangle = \|x\|^2,$$

and hence V preserves the norm on $(\ker V^*V)^{\perp}$. Moreover, if $V^*Vx = 0$, then $0 = \langle V^*Vx, x\rangle = \|Vx\|^2$ and consequently $\ker V^*V = \ker V$. Therefore, V is a partial isometry, and hence (i) and (iii) are equivalent.

Similarly, we have the equivalence of (ii) and (iv).

Moreover, if V^*V is a projection, then $(VV^*)^2 = VV^*VV^* = VV^*$, since $V(V^*V) = V$. Therefore, VV^* is a projection, which completes the proof. □

We now obtain the polar decomposition for an operator.

Theorem 1.13 *If A is an operator on a Hilbert space H, then there exists a positive operator P and a partial isometry V such that $A = VP$. Moreover, V and P are unique if* $\ker P = \ker V$.

Proof. If we set $P = |A|$, then

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle A^*Ax, x\rangle = \|Ax\|^2 \quad \text{for } x \in H.$$

Thus, if we define $\tilde{V}$ on ran P such that $\tilde{V}Px = Ax$, then $\tilde{V}$ is well defined and is isometric. Hence, $\tilde{V}$ can be extended uniquely to an isometry from $\mathrm{clos}(\mathrm{ran}P)$ to H. If we further extend $\tilde{V}$ to H by defining it to be the zero operator on $(\mathrm{ran}P)^\perp$, then the extended extended operator V is a partial isometry satisfying $A = VP$ and $\ker V = (\mathrm{ran}P)^\perp = \ker P$ by Lemma 1.3.

We next consider uniqueness. Suppose $A = WQ$, where W is a partial isometry, Q is a positive operator, and $\ker W = \ker Q$. Then $P^2 = A^*A = QW^*WQ = Q^2$, since W^*W is the projection onto

$$(\ker W)^\perp = (\ker Q)^\perp = \mathrm{clos}(\mathrm{ran}\ Q).$$

Thus, by the uniqueness of the square root, we have $P = Q$ and hence $WP = VP$. Therefore, $W = V$ on ran P. But

$$(\mathrm{ran}\ P)^\perp = \ker P = \ker W = \ker V$$

and hence $W = V$ on $(\mathrm{ran}\ P)^\perp$. Therefore, $V = W$ and the proof is complete. □

Corollary 1.2 *If A is an operator on a Hilbert space H, then there exists a positive operator Q and a partial isometry W such that $A = QW$. Moreover, W and Q are unique if* $\mathrm{ran}\ Q = (\ker Q)^\perp$.

Proof. By Theorem 1.13, we obtain a partial isometry V and a positive operator P such that $A^* = VP$. Taking adjoints we have $A = PV^*$, which is the form that we desire with $W = V^*$ and $Q = P$. Moreover, the uniqueness also follows from Theorem 1.13 since $\mathrm{ran}\ W = (\ker Q)^\perp$ if and only if

$$\ker V = \ker W^* = (\mathrm{ran}\ W)^\perp = (\ker Q)^{\perp\perp} = \ker P.$$

□

1.4 Notes

For our exposition we have used [276], [45], [143], [18].

Chapter 2

Kantorovich Inequality and Mond-Pečarić Method

This chapter tells the history of the Kantorovich inequality, and describes how the Kantorovich inequality has developed in the field of operator inequalities. In such context, so called "the Mond-Pečarić method" for convex functions established by Mond and Pečarić has outlined a more complete picture of that inequality in the field of operator inequalities.

2.1 History

The story of the Kantorovich inequality is a very interesting example how a mathematician creates mathematics. It provides a deep insight into how a principle raised from the Kantorovich inequality has developed in the field of operator inequalities on a Hilbert space, and perhaps, more importantly, it has initiated a new way of thinking and new methods in operator theory, noncommutative differential geometry, quantum information theory and noncommutative probability theory. We call this principle *the Mond-Pečarić method for convex functions*.

In 1959, Greub and Rheinboldt published the celebrated paper [132]. It is just the birth of the Kantorovich inequality. They stated that Kantorovich proved the following inequality.

Theorem K1 *If the sequence* $\{\gamma_k\}$ $(k = 1, 2, \cdots)$ *of real numbers has the property*

$$0 < m \le \gamma_k \le M$$

and $\{\xi_k\}$ $(k = 1, 2, \cdots)$ *denotes another sequence with* $\sum_{k=1}^{\infty} \xi_k^2 < \infty$*, then the inequality*

$$\sum_{k=1}^{\infty} \gamma_k \xi_k^2 \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \xi_k^2 \le \frac{(M+m)^2}{4Mm} \left[\sum_{k=1}^{\infty} \xi_k^2 \right]^2 \tag{2.1}$$

holds.

It seems to be the first paper which introduced (2.1) to the world of mathematics. Moreover, they say that Kantorovich pointed out that (2.1) is a special case of the following inequality enunciated by G. Pólya and G. Szegö [253].

Theorem PS *If real numbers* a_k *and* b_k $(k = 1, \cdots, n)$ *fulfill the conditions*

$$0 < m_1 \le a_k \le M_1 \qquad \text{and} \qquad 0 < m_2 \le b_k \le M_2$$

respectively, then

$$1 \le \frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{\left[\sum_{k=1}^{n} a_k b_k\right]^2} \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{2.2}$$

To understand (2.1) in Theorem K1 well, if we put $\xi_k = 1/\sqrt{n}$ for $k = 1, \cdots, n$, then (2.1) implies

$$\frac{\gamma_1 + \cdots + \gamma_n}{n} \cdot \frac{\gamma_1^{-1} + \cdots + \gamma_n^{-1}}{n} \le \frac{(M+m)^2}{4Mm}. \tag{2.3}$$

Summing up, whenever γ_k's move in the closed interval $[m, M]$, the left-hand side of (2.3) does not absolutely exceed the constant $\frac{(M+m)^2}{4Mm}$. At present, the constant $\frac{(M+m)^2}{4Mm}$ is called the *Kantorovich constant*.

Greub and Rheinboldt moreover went ahead with the ideas of Kantorovich and proved the following theorem as a generalization of the Kantorovich inequality.

Theorem K2 *Given a self-adjoint operator* A *on a Hilbert space* H*. If* A *fulfills the condition*

$$mI_H \le A \le MI_H \quad \text{for some scalars } 0 < m \le M,$$

then

$$\langle x, x\rangle^2 \le \langle Ax, x\rangle \langle A^{-1}x, x\rangle \le \frac{(M+m)^2}{4Mm} \langle x, x\rangle^2 \tag{2.4}$$

for all $x \in H$.

Though this formulation is very simple, how to generalize (2.1) might be not plain. In the case that A is matrix, then (2.4) can be expressed as follows: Put

$$A = \begin{pmatrix} \gamma_1 & & & & 0 \\ & \gamma_2 & & & \\ & & \ddots & & \\ & & & \gamma_n & \\ 0 & & & & \ddots \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \\ \vdots \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} \gamma_1^{-1} & & & & 0 \\ & \gamma_2^{-1} & & & \\ & & \ddots & & \\ & & & \gamma_n^{-1} & \\ 0 & & & & \ddots \end{pmatrix}$$

and we get

$$\langle Ax,x\rangle = \sum_{k-1}^{\infty} \gamma_k \xi_k^2 \quad \text{and} \quad \langle A^{-1}x,x\rangle = \sum_{k=1}^{\infty} \gamma_k^{-1} \xi_k^2.$$

We shall agree that (2.4) is called a generalization of the Kantorovich inequality (2.1).

Though Greub and Rheinboldt carefully cite the Kantorovich inequality, they do not tell anything about his motivation for considering the inequality (2.1). What is his motive for considering (2.1)? Thus, we shall attempt to investigate Kantorovich's original paper in this occasion. It is written in Russian and very old. We read the original paper in an English translation [156]. It seems that he was interested in the mathematical formulation of economics, as he provided a detailed commentary on how to carry out mathematical analysis in economic activities. Now, when we read [156] slowly and carefully, we find the inequality (2.1) in question, in the middle of the paper [156].

Lemma K *The inequality*

$$\sum_k \gamma_k u_k^2 \sum_k \gamma_k^{-1} u_k^2 \le \frac{1}{4}\left[\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right]^2 \left(\sum_k u_k^2\right)^2 \tag{2.5}$$

holds, m and M being the bounds of the numbers γ_k

$$0 < m \le \gamma_k \le M.$$

The coefficient in the right-hand side of (2.5) seems to be different from the one in (2.1). However, since

$$\frac{1}{4}\left[\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right]^2 = \frac{1}{4}\left[\frac{M+m}{\sqrt{Mm}}\right]^2 = \frac{(M+m)^2}{4Mm},$$

the constant of (2.5) coincides with one of (2.1). Following Kantorovich's original paper, we know that Kantorovich represents an upper bound as (2.5). Therefore the Kantorovich constant $\frac{(M+m)^2}{4Mm}$ is deformed by Greub and Rheinboldt. Examining the history of mathematics a little more, Henrici [141] pointed out that in the case of equal weights, the inequality (2.3) is due to Schweitzer [258] in 1914. How Kantorovich proved the inequality (2.5) in Lemma K is a very interesting matter:

Proof of Lemma K. We may prove it in the case of finite sums $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$ and $\sum_{k=1}^n u_k^2 = 1$. We shall seek the maximum of

$$G = \sigma\tilde{\sigma} = \left(\sum_{k=1}^n \gamma_k u_k^2\right)\left(\sum_{k=1}^n \frac{1}{\gamma_k} u_k^2\right)$$

under the condition that $\sum_{k=1}^n u_k^2 = 1$. By using the method of Lagrange multipliers, if we equate to zero the derivatives of the function

$$F = G - \lambda\left(\sum_{k=1}^n u_k^2 - 1\right),$$

then we have

$$\frac{1}{2}\frac{\partial F}{\partial u_s} = \sigma\frac{1}{\gamma_s}u_s + \tilde{\sigma}\gamma_s u_s - \lambda u_s = 0, \quad \text{i.e. } u_s(\sigma + \tilde{\sigma}\gamma_s^2 - \lambda\gamma_s) = 0.$$

The second factor in the last expression, being a polynomial of the second degree in γ_s, can reduce to zero at not more than two values of s; let these be $s = k, l$. For the remaining values of s, u_s must be zero. But then

$$\begin{aligned} G_{\max} &= \left(\gamma_k u_k^2 + \gamma_l u_l^2\right)\left(\frac{1}{\gamma_k}u_k^2 + \frac{1}{\gamma_l}u_l^2\right) \\ &= \frac{1}{4}\left[\sqrt{\frac{\gamma_k}{\gamma_l}} + \sqrt{\frac{\gamma_l}{\gamma_k}}\right]^2 (u_k^2 + u_l^2)^2 - \frac{1}{4}\left[\sqrt{\frac{\gamma_k}{\gamma_l}} + \sqrt{\frac{\gamma_l}{\gamma_k}}\right]^2 (u_k^2 - u_l^2)^2 \\ &\le \frac{1}{4}\left[\sqrt{\frac{\gamma_k}{\gamma_l}} + \sqrt{\frac{\gamma_l}{\gamma_k}}\right]^2 \le \frac{1}{4}\left[\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right]^2. \end{aligned}$$

□

Why does Kantorovich need the inequality (2.1)? If we only read the paper due to Greub and Rheinboldt, we probably cannot fully understand those circumstances. However, having thoroughly read [156], we are able to explain the necessity of the Kantorovich inequality.

Kantorovich says that as is generally known, a significant part of the problems of mathematical physics – the majority of the linear problems of analysis – may be reduced to a problem of the extremum of quadratic functionals. This fact may be utilized, on the one hand for different theoretical investigations relating to these problems. On the other hand, it serves as a basis for direct methods of solving the problems named. A certain method of successive approximations for the solution of problems concerning the minimum of quadratic functionals, and of the linear problems connected with them, is elaborated – the method of steepest descent.

Let H be a real Hilbert space and A a self-adjoint (bounded linear) operator on H such that $mI_H \le A \le MI_H$ for some scalars $0 < m \le M$.

We shall consider the method of steepest descent as it applies to the solution of the equation

$$L(x) = Ax - y = 0, \tag{2.6}$$

where x and y are in H. We introduce the quadratic functional

$$H(x) = \langle Ax, x\rangle - 2\langle y, x\rangle. \tag{2.7}$$

For a given $y \in H$, a vector $x_0 \in H$ is the solution of $L(x) = 0$ if and only if $x_0 \in H$ attains the minimum of $H(x)$.

Indeed, suppose that $x \in H$ satisfies $H(x) = \min_{u\in H} H(u)$. Then for each nonzero $z \in H$ and a real parameter $\alpha \in \mathbb{R}$, it follows that

$$H(x+\alpha z) - H(x) \geq 0$$

and this implies

$$\begin{aligned} H(x+\alpha z) - H(x) &= \langle Ax + \alpha Az, x + \alpha z\rangle - 2\langle y, x+\alpha z\rangle - H(x) \\ &= \alpha\left[\langle Ax, z\rangle + \langle Az, x\rangle\right] + \alpha^2\langle Az, z\rangle - 2\alpha\langle y, z\rangle \\ &= 2\alpha\langle Ax - y, z\rangle + \alpha^2\langle Az, z\rangle \geq 0. \end{aligned}$$

Since A is positive invertible, we have $\langle Az, z\rangle > 0$. Since the inequality above holds for all $\alpha \in \mathbb{R}$, we get $(Ax - y, z) = 0$ for all nonzero $z \in H$. Therefore we have $Ax - y = 0$ and hence $x \in H$ is the solution of $L(x) = 0$.

Conversely, suppose that $x \in H$ is the solution of $L(x) = Ax - y = 0$. Then

$$H(x+z) - H(x) = \langle Az, z\rangle + 2\langle Ax - y, z\rangle = \langle Az, z\rangle > 0 \tag{2.8}$$

for all nonzero $z \in H$. For each $y \in H$, if we put $z = y - x$ in (2.8), then we have $H(y) \geq H(x)$ and this implies $H(x) = \min_{y\in H} H(y)$.

In this way, if the problem of solving an equation (2.6) reduces to the problem of seeking the minimum of the functional (2.7), then this fact is named the variational principle of the equation.

In seeking the minimum of a functional (2.7) we shall employ the method of steepest descent. Now, we consider the following three procedures (0), (1) and (2):

(0) For a given initial vector $x_0 \in H$, we find a sequence $\{x_n\} \subset H$ such that

$$H(x_0) > H(x_1) > \cdots > H(x_n) > \cdots \to \min_{u\in H} H(u) = H(x).$$

(1) By induction, we construct a sequence $\{x_n\} \subset H$ such that

$$x_{n+1} = x_n + \alpha_n z_n$$

for $\alpha_n \in \mathbb{R}$ and $z_n \in H$.

(2) Moreover, we choose $\alpha_n \in \mathbb{R}$ such that

$$H(x_n + \alpha_n z_n) = \min_{t\in\mathbb{R}} H(x_n + t z_n). \tag{2.9}$$

The following lemma shows that the condition (0) implies the convergence of $\{x_n\}$.

Lemma 2.1 *Let x be the solution of $L(x) = Ax - y = 0$. If a sequence $\{x_n\}$ satisfies*

$$H(x_0) > H(x_1) > \cdots > H(x_n) > \cdots \to \min_{u \in H} H(u) = H(x),$$

then $x_n \to x$ as $n \to \infty$.

Proof.

$$\begin{aligned} H(x_n) - H(x) &= \langle Ax_n, x_n \rangle - 2\langle y, x_n \rangle - \langle Ax, x \rangle + 2\langle y, x \rangle \\ &= 2\langle Ax - y, x_n - x \rangle + \langle A(x_n - x), x_n - x \rangle \\ &= \langle A(x_n - x), x_n - x \rangle \geq m\|x_n - x\|^2, \end{aligned}$$

because $m\langle z, z \rangle \leq \langle Az, z \rangle \leq M\langle z, z \rangle$ for every $z \in H$ by the assumption. Therefore $\lim_{n\to\infty} H(x_n) = H(x)$ implies $\lim_{n\to\infty} x_n = x$. □

The following lemma determines the form of α_n.

Lemma 2.2 *If* (2.9) *holds, then*

$$\alpha_n = \frac{\langle z_n, z_n \rangle}{\langle Az_n, z_n \rangle}$$

where $z_n = y - Ax_n$.

Proof.

$$\begin{aligned} H(x_n + tz_n) &= \langle Az_n, z_n \rangle t^2 + 2(\langle Ax_n, z_n \rangle - \langle y, z_n \rangle)t + H(x_n) \\ &= \langle Az_n, z_n \rangle t^2 + 2\langle z_n, z_n \rangle t + H(x_n) \\ &= \langle Az_n, z_n \rangle \left(t - \frac{\langle z_n, z_n \rangle}{\langle Az_n, z_n \rangle} \right)^2 - \frac{\langle z_n, z_n \rangle^2}{\langle Az_n, z_n \rangle} + H(x_n) \end{aligned}$$

Therefore, $t = \frac{\langle z_n, z_n \rangle}{\langle Az_n, z_n \rangle}$ attains the minimum of $H(x_n + tz_n)$. □

By the proof of Lemma 2.2, we have

$$H(x_{n+1}) = H(x_n) - \frac{\langle z_n, z_n \rangle^2}{\langle Az_n, z_n \rangle} < H(x_n)$$

and hence we have

$$H(x_0) > H(x_1) > \cdots > H(x_n) > \cdots.$$

Theorem K4 *The successive approximations $\{x_n\} \subset H$ constructed by the method of steepest descent converge strongly to the solution of the equation* (2.6) *with the speed of a geometrical progression.*

Proof. Let x^* be the solution of equation (2.6) and $\Delta_n H = H(x_n) - H(x^*)$. It is obtained that the change $\Delta_n H$ of H in passing from x^* to x_n is

$$\Delta_n H = H(x_n) - H(x^*) = \langle A(x^* - x_n), x^* - x_n \rangle.$$

Also, since

$$z_n = y - Ax_n$$

and

$$z_{n+1} = y - Ax_{n+1} = z_n - \alpha_n A z_n,$$

it follows that

$$\Delta_n H = \langle A(x_n - x^*), x_n - x^* \rangle = \langle A^{-1} z_n, z_n \rangle$$

and

$$\Delta_{n+1} H = \langle A(x_{n+1} - x^*), x_{n+1} - x^* \rangle = \Delta_n H - 2\alpha_n \langle z_n, z_n \rangle + \alpha_n^2 \langle A z_n, z_n \rangle.$$

By the definition of α_n, we have

$$\begin{aligned} \frac{\Delta_n H - \Delta_{n+1} H}{\Delta_n H} &= \frac{2\alpha_n \langle z_n, z_n \rangle - \alpha_n^2 \langle A z_n, z_n \rangle}{\langle A^{-1} z_n, z_n \rangle} \\ &= \frac{\langle z_n, z_n \rangle^2}{\langle A z_n, z_n \rangle \langle A^{-1} z_n, z_n \rangle} \end{aligned} \tag{2.10}$$

We notice the form of a generalization of the Kantorovich inequality due to Greub-Rheinboldt in the last expression of (2.10).

For the estimation of this ratio let us make use of the spectral decomposition of an operator A:

$$A = \int_m^M \lambda \, de_\lambda \quad \text{and} \quad \langle A z_1, z_1 \rangle = \int_m^M \lambda \, \mathrm{d}\langle e_\lambda z_1, z_1 \rangle = \lim \sum \lambda \langle \Delta e_\lambda z_1, z_1 \rangle; \tag{2.11}$$

analogously

$$\langle z_1, z_1 \rangle = \lim \sum \langle \Delta e_\lambda z_1, z_1 \rangle \quad \text{and} \quad \langle A^{-1} z_1, z_1 \rangle = \lim \sum \frac{1}{\lambda} \langle \Delta e_\lambda z_1, z_1 \rangle. \tag{2.12}$$

Replacing in expression (2.10) the inner product by their approximate value as given by (2.11) and (2.12), we have

$$\begin{aligned} \frac{\Delta_n H - \Delta_{n+1} H}{\Delta_n H} &= \frac{[\sum \langle \Delta e_\lambda z_1, z_1 \rangle]^2}{\sum \lambda \langle \Delta e_\lambda z_1, z_1 \rangle \sum \frac{1}{\lambda} \langle \Delta e_\lambda z_1, z_1 \rangle} \\ &\geq \frac{4Mm}{(M+m)^2} > 0. \end{aligned}$$

The Kantorovich inequality is utilized here to estimate a lower bound!

The approximate equality here is correct with as small an error as one pleases, and we have therefore an exact inequality

$$\frac{\Delta_n H - \Delta_{n+1} H}{\Delta_n H} \geq \frac{4Mm}{(M+m)^2},$$

whence

$$\Delta_{n+1} H \leq \left(1 - \frac{4Mm}{(M+m)^2}\right) \Delta_n H = \left(\frac{M-m}{M+m}\right)^2 \Delta_n H.$$

Since $0 \leq \frac{M-m}{M+m} < 1$, for a given initial vector x_0, we have

$$\lim_{n\to\infty} \Delta_n H = 0$$

so that $\lim_{n\to\infty} H(x_n) = H(x^*)$. By Lemma 2.1, we have $x_n \to x^*$ as $n \to \infty$ and this proves the assertion. □

The rapidity of convergence of the process is of the order of a geometric progression with ratio $q = (M-m)/(M+m)$.

It is surprising that the Kantorovich inequality is utilized in the linear problems of analysis. We cannot understand this fact by reading [132] only. Also, as mentioned above, we think that Kantorovich proved the following form: If an operator A on H is positive such that $mI_H \leq A \leq MI_H$ for some scalars $0 < m < M$, then

$$\frac{\langle x,x\rangle^2}{\langle Ax,x\rangle\langle A^{-1}x,x\rangle} \geq \frac{4}{\left[\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right]^2} \tag{2.13}$$

holds for every nonzero vector x in H.

Namely, the Kantorovich inequality is not only the form (2.1) shown in Lemma K, but also the form (2.13) of the operator version.

Now, the theorem denoted by K2 is a generalization of the Kantorovich inequality in the operator form, as it was derived by Greub and Rheinboldt. In fact, we easily see that (2.13) implies Theorem K2. Therefore, one could say that Kantorovich proved Theorem K2 in a certain sense. At this point, it is suitable to cite a relevant part of [132]:

> The subject of this paper is the proof of a generalized form of the inequality for linear, bounded and self-adjoint operators in Hilbert space. This generalized Kantorovich inequality proves to be equivalent to a similarly generalized form of the inequality which we shall call the generalized Pólya-Szegö inequality. Our generalized Kantorovich inequality is already implicitly contained in the paper of L.V.Kantorovich. However, its proof there involves the use of the theory of spectral decomposition for the operators in question. The proof we shall present here will proceed in a considerable simpler way.

Hence, from the underlined sentence we learn that the proof of Theorem K2 was essentially contained in [156]. Furthermore, we see that Greub and Rheinboldt prefer to avoid the spectral decomposition theorem in the proof, as they believe their own proof to be considerably simpler.

However, it turned out that their method of proof had a deep significance for mathematics. The impact of Theorem K2 could be compared to spreading of shock waves around the world of mathematics. Thus we present the proof of which Greub and Rheinboldt say that is simpler.

Proof of Theorem K2. The left hand side of the inequality follows directly from Schwarz's inequality

$$\begin{aligned}\langle x,x\rangle^2 &= \langle A^{1/2}x, A^{-1/2}x\rangle^2 \leq \langle A^{1/2}x, A^{1/2}x\rangle\langle A^{-1/2}x, A^{-1/2}x\rangle \\ &= \langle Ax,x\rangle\langle A^{-1}x,x\rangle.\end{aligned}$$

We shall first prove the right hand side of (2.4) for finite dimensional space H. Then we will show that the proof for the general case can be reduced to that of the finite dimensional case.

Suppose that H is a finite dimensional space. Then the unit sphere $S \subset H$ is compact. Hence, considered on S, the continuous functional

$$f(x) = \frac{\langle Ax,x\rangle\langle A^{-1}x,x\rangle}{\langle x,x\rangle^2}$$

attains its maximum at a certain point, say $x_0 \in S$, i.e.

$$f(x_0) = \max_{x\in S} f(x) = \langle Ax_0,x_0\rangle\langle A^{-1}x_0,x_0\rangle.$$

With a fixed vector $y \in H$ and the real parameter t ($|t| < 1$) we consider the real valued function

$$g(t) = f(x_0 + ty).$$

This function $g(t)$ has a relative maximum at $t = 0$ and therefore we must necessarily have $g'(0) = 0$. Using the self-adjointness of A and A^{-1} we find

$$g'(0) = 2\langle Ax_0,y\rangle\langle A^{-1}x_0,x_0\rangle + 2\langle A^{-1}x_0,y\rangle\langle Ax_0,x_0\rangle - 4f(x_0)\langle x_0,y\rangle = 0$$

and thus

$$\langle \gamma Ax_0 + \mu A^{-1}x_0 - x_0, y\rangle = 0$$

holds for all $y \in H$, where

$$\gamma = \frac{1}{2\langle Ax_0,x_0\rangle} \quad \text{and} \quad \mu = \frac{1}{2\langle A^{-1}x_0,x_0\rangle}.$$

Consequently

$$x_0 = \gamma Ax_0 + \mu A^{-1}x_0.$$

Applying A and A^{-1} successively to this equation we find that

$$Ax_0 = \gamma A^2x_0 + \mu x_0 \quad \text{and} \quad A^{-1}x_0 = \gamma x_0 + \mu A^{-2}x_0$$

or

$$\left(A - \frac{1}{2\gamma}I_H\right)^2 x_0 = \frac{1-4\gamma\mu}{4\gamma^2}x_0 \text{ and } \left(A^{-1} - \frac{1}{2\mu}I_H\right)^2 x_0 = \frac{1-4\gamma\mu}{4\mu^2}x_0.$$

Taking into account the assumption $0 < mI_H \le A \le MI_H$, we have

$$4\gamma\mu\frac{m}{M} \le \left(1 + (1-4\gamma\mu)^{1/2}\right)^2 \le 4\gamma\mu\frac{M}{m}.$$

It follows

$$\left[4\gamma\mu\left(\frac{m}{M}+1\right)-2\right]^2 \le 4(1-4\gamma\mu) \le \left[4\gamma\mu\left(\frac{M}{m}+1\right)-2\right]^2$$

or

$$\frac{\gamma\mu}{M^2}\left[4\gamma\mu(M+m)^2-4mM\right]\le 0\le\frac{\gamma\mu}{m^2}\left[4\gamma\mu(M+m)^2-4mM\right]$$

and therefore

$$4\gamma\mu(M+m)^2-4mM=0.$$

On the other hand, since

$$4\gamma\mu=\frac{1}{\langle Ax_0,x_0\rangle\langle A^{-1}x_0,x_0\rangle},$$

we finally have

$$\langle Ax_0,x_0\rangle\langle A^{-1}x_0,x_0\rangle=\frac{(M+m)^2}{4Mm}, \tag{2.14}$$

which was to be proved. (2.14) shows furthermore that (at least in the finite dimensional case) the upper bound in (2.4) can not be improved.

We now remove the restriction of the finite-dimensionality of H. Let x_0 be a fixed vector of H and let $H_0\subset H$ be a finite dimensional subspace of H which contains three vectors x_0, Ax_0 and $A^{-1}x_0$. We denote by P the projection of H onto H_0. For the operator $B=PA$, we have $B(H_0)\subset H_0$ and

$$\langle Bx,y\rangle=\langle PAx,y\rangle=\langle PAPx,y\rangle=\langle x,PAPy\rangle=\langle x,By\rangle$$

for all $x,y\in H_0$. Hence, B is a self-adjoint operator on the space H_0. Furthermore, we find for $x\in H_0$

$$\langle Bx,x\rangle=\langle PAx,x\rangle=\langle Ax,Px\rangle=\langle Ax,x\rangle$$

and therefore in H_0

$$0<mI_{H_0}\le m'I_{H_0}\le B\le M'I_{H_0}\le MI_{H_0} \tag{2.15}$$

where

$$m'=\inf_{x\in H_0}\frac{\langle Bx,x\rangle}{\langle x,x\rangle}\qquad\text{and}\qquad M'=\sup_{x\in H_0}\frac{\langle Bx,x\rangle}{\langle x,x\rangle}.$$

Hence, we can apply the first part of the proof to the operator B in the finite dimensional space H_0. By doing that we obtain for all $x\in H_0$

$$\frac{\langle Bx,x\rangle\langle B^{-1}x,x\rangle}{\langle x,x\rangle^2}\le\frac{(M'+m')^2}{4m'M'}=\frac{1}{4}\left(\frac{M'}{m'}+\frac{m'}{M'}\right)+\frac{1}{2}. \tag{2.16}$$

From (2.15) we conclude that

$$1\le\frac{M'}{m'}\le\frac{M}{m}\qquad\text{and}\qquad\frac{M'}{m'}+\frac{m'}{M'}\le\frac{M}{m}+\frac{m}{M}. \tag{2.17}$$

This last inequality is a result of the fact that for $u\ge 1$ the function $f(u)=u+1/u$ is monotonically increasing. (2.16) and (2.17) together yield

$$\frac{\langle Bx,x\rangle\langle B^{-1}x,x\rangle}{\langle x,x\rangle^2}\le\frac{1}{4}\left(\frac{M}{m}+\frac{m}{M}\right)+\frac{1}{2}=\frac{(M+m)^2}{4mM}$$

for all $x \in H_0$. Since H_0 contains x_0, Ax_0 and $A^{-1}x_0$, we find

$$Bx_0 = PAx_0 = Ax_0 \qquad \text{and} \qquad x_0 = Px_0 = PAA^{-1}x_0 = BA^{-1}x_0.$$

The last relation implies $B^{-1}x_0 = A^{-1}x_0$ when one considers that the existence of B^{-1} in H_0 is a direct consequence of (2.15). Substituting we obtain finally

$$\langle Ax_0, x_0\rangle \langle A^{-1}x_0, x_0\rangle \leq \frac{(M+m)^2}{4mM}\langle x_0, x_0\rangle^2$$

Since x_0 was arbitrary the theorem is hereby completely proved. □

Moreover, they showed the generalized Pólya-Szegö inequality, which is equivalent to the Kantorovich inequality:

Theorem 2.1 *Let A and B be commuting self-adjoint operators on a Hilbert space H such that*

$$0 < m_1 I_H \leq A \leq M_1 I_H \qquad \textit{and} \qquad 0 < m_2 I_H \leq B \leq M_2 I_H.$$

Then

$$\langle Ax, Ax\rangle \langle Bx, Bx\rangle \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2}\langle Ax, Bx\rangle^2$$

for all $x \in H$.

Proof. It is rather obvious that the Kantorovich inequality is contained in Theorem 2.1. In fact, let C be any given self-adjoint operator with

$$0 < mI_H \leq C \leq MI_H.$$

We set $A = C^{1/2}$ and $B = (C^{-1})^{1/2}$. Since

$$0 < m^{1/2}I_H \leq A \leq M^{1/2}I_H \qquad \text{and} \qquad 0 < (M^{-1})^{1/2}I_H \leq B \leq (m^{-1})^{1/2}I_H,$$

it follows immediately from Theorem 2.1 that

$$\frac{\langle Cx, x\rangle \langle C^{-1}x, x\rangle}{\langle x, x\rangle^2} = \frac{\langle Ax, Ax\rangle \langle Bx, Bx\rangle}{\langle Ax, Bx\rangle^2} \leq \frac{(M+m)^2}{4mm}$$

for all $x \in H$ and this is the statement of the Kantorovich inequality.

Next, we show that Theorem 2.1 is a consequence of Theorem K2.

From the commutativity of A and B, for the self-adjoint operator $C = AB^{-1}$ we have

$$0 < \frac{m_1}{M_2}I_H \leq C \leq \frac{M_1}{m_2}I_H.$$

Therefore, it follows from Theorem K2 that

$$\frac{\langle Cx, x\rangle \langle C^{-1}x, x\rangle}{\langle x, x\rangle^2} \leq \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2}$$

for all $x \in H$. Put $x = (AB)^{1/2}y$, then we obtain $\langle Cx,x\rangle = \langle Ay,Ay\rangle$, $\langle C^{-1}x,x\rangle = \langle By,By\rangle$ and $\langle x,x\rangle = \langle Ay,By\rangle$. Substituting these relations, we get the statement of Theorem 2.1. □

The proof by Greub and Rheinboldt is very long, spanning over approximately five pages. We can feel the strictness of their proof, but, in contrast, Kantorovich's proof is simple and only half a page long. However, it was the formulation by Greub and Rheinboldt that brought the first wave of excitement into the world of mathematics. Owing to Greub and Rheinboldt, the work of Kantorovich has become an object of research in mathematics, in operator theory in particular. In their own words, their proof is simple. But, it is a proof on a grand scale, unexpected and fascinating. Based on a beautiful relation, this simple formulation may strike a chord in the heart of a mathematician. Many mathematicians concentrated their energies on the generalization of the Kantorovich inequality and on searching for an even simpler proof.

2.2 Generalizations and improvements

In 1960, one year after the publication of [132], Strang [272] shows the following generalization of the Kantorovich inequality for an arbitrary operator without conditions such as self-adjoiness and positivity.

Theorem 2.2 *If T is an arbitrary invertible operator on H, and $\|T\| = M$, $\|T^{-1}\|^{-1} = m$, then*

$$|\langle Tx,y\rangle\langle x,T^{-1}y\rangle| \le \frac{(M+m)^2}{4Mm}\langle x,x\rangle\langle y,y\rangle \qquad \textit{for all } x,y \in H.$$

Furthermore, this bound is the best possible.

Proof. We consider the polar decomposition of T. Let $A = (T^*T)^{1/2}$. Then $U = TA^{-1}$ is unitary, and

$$\begin{aligned} |\langle Tx,y\rangle\langle x,T^{-1}y\rangle| &= |\langle UAx,x\rangle\langle x,A^{-1}U^{-1}y\rangle| = |\langle Ax,U^*y\rangle\langle A^{-1}x,U^*y\rangle| \\ &\le \left[\langle Ax,x\rangle\langle AU^*y,U^*y\rangle\langle A^{-1}x,x\rangle\langle A^{-1}U^*y,U^*y\rangle\right]^{1/2} \end{aligned} \tag{2.18}$$

by generalized Schwarz's inequality (Theorem 1.10). Since $\|A\| = \|(T^*T)^{1/2}\|$ $= \|T\| = M$ and $\|A^{-1}\|^{-1} = \|T^{-1}\|^{-1} = m$, it follows that $mI_H \le A \le MI_H$. Therefore, by (2.4) in Theorem K2, we have

$$\begin{aligned} \text{RHS in (2.18)} &\le \left(\frac{(M+m)^2}{4Mm}\langle x,x\rangle^2 \cdot \frac{(M+m)^2}{4Mm}\langle U^*y,U^*y\rangle^2\right)^{1/2} \\ &= \frac{(M+m)^2}{4Mm}\langle x,x\rangle\langle y,y\rangle, \end{aligned}$$

by using $\langle U^*y,U^*y\rangle = \langle y,y\rangle$.

If H is finite dimensional, the bound is attained for $x = U^*y = u + v$, where u and v are unit eigenvectors of A corresponding to eigenvalues m and M. In a general case, the bound need not be attained. But a sequence $x_n = U^*y_n = u_n + v_n$, where $\|u_n\| = \|v_n\|$, $(e(m+1/n) - e(m-0))u_n = u_n$, $(e(M+0) - e(M-1/n))v_n = v_n$ shows on calculation that the bound is best possible. □

Also, Schopf [257] considered a generalization of the power in the Kantorovich inequality. Moving to the year 1996, there is the following extension due to Spain [270] which is totally different from the Kantorovich inequality. But it is surely an extension. It does not assume positivity, either. It is slightly long, but we will quote it:

The Kantorovich inequality says that if A is a positive operator on a Hilbert space H such that $mI_H \le A \le MI_H$ for some scalars $0 < m \le M$, then

$$4mM\langle A^{-1}x,x\rangle \le (m+M)^2 \frac{\|x\|^4}{\langle Ax,x\rangle}$$

holds for every vector x in H. If we replace x by $A^{\frac{1}{2}}x$, then

$$4mM\langle x,x\rangle \le (m+M)^2 \frac{\|A^{\frac{1}{2}}x\|^2}{\langle A^2x,x\rangle}.$$

This inequality may be viewed as a conversion of the special case

$$\langle Ax,x\rangle \le \|Ax\|\,\|x\|$$

of the Cauchy-Schwarz inequality, for it is equivalent to the inequality

$$2\sqrt{mM}\|Ax\|\,\|x\| \le (m+M)\langle Ax,x\rangle.$$

The methods of operator and spectral theory allow one to generalize the inequality to a wide class of operators on a Hilbert space.

Let Γ be any nonzero complex number, let $R = |\Gamma|$, and let $0 \le r \le R$.

Theorem 2.3 *Let A be an operator on H such that $\left|A - \Gamma[A]\right|^2 \le r^2[A]$, where $[A]$ is the range projection of A. Let $u \in B(K,H)$ be an operator such that $u^*[A]u$ is a projection. Then*

$$(R^2 - r^2)u^*A^*Au \le R^2(u^*A^*u)(u^*Au).$$

Proof. Since $u^*[A]u$ is a projection, we have

$$\begin{aligned}&\left|(R^2 - r^2)u^*[A]u - \overline{\Gamma}u^*Au\right|^2\\ &= (R^2 - r^2)^2u^*[A]u - (R^2 - r^2)\{\overline{\Gamma}u^*Au + \Gamma u^*A^*u\} + R^2(u^*A^*u)(u^*Au),\end{aligned}$$

while

$$\begin{aligned}&u^*\left(r^2[A] - \left|A - \Gamma[A]\right|^2\right)u\\ &= -(R^2 - r^2)u^*[A]u - u^*A^*Au + \overline{\Gamma}u^*Au + \Gamma u^*A^*u,\end{aligned}$$

and hence

$$\begin{aligned}&R^2u^*A^*uu^*Au-(R^2-r^2)u^*A^*Au\\&=\left|(R^2-r^2)u^*[A]u-\overline{\Gamma}u^*Au\right|^2+u^*\Big(r^2[A]-\left|A-\Gamma[A]\right|^2\Big)u.\end{aligned}$$

By the assumption of $|A-\Gamma[A]|^2\le r^2[A]$, we have

$$R^2(u^*A^*u)(u^*Au)-(R^2-r^2)u^*A^*Au\ge 0.$$

□

Corollary 2.1 *Let A be a positive operator on H such that A is invertible on its range, let $m=\min\mathsf{Sp}(A)\backslash\{0\}$ and $M=\max\mathsf{Sp}(A)=\|A\|$. Let $u\in B(K,H)$ be an operator such that $u^*[A]u$ is a projection. Then*

$$4Mmu^*A^2u\le(M+m)^2(u^*Au)^2.$$

Proof. In the situation of Theorem 2.3, we have

$$R=\Gamma=\frac{M+m}{2}\qquad\text{and}\qquad r=\frac{M-m}{2}.$$

By the assumption of A, it follows that

$$m[A]\le A\le M[A]$$

and hence $|A-\Gamma[A]|^2\le r^2[A]$. Therefore Corollary 2.1 follow from Theorem 2.3. □

Theorem 2.4 *Let A be an operator on H such that $|A-\Gamma[A]|^2\le r^2[A]$. Then*

$$(R^2-r^2)^{1/2}\|Ax\|\left\|[A]x\right\|\le R|\langle Ax,x\rangle|,\qquad x\in H.$$

If A is positive with $\mathsf{Sp}(A)\backslash\{0\}\subset[m,M]$ $(0<m<M)$, then

$$2\sqrt{Mm}\|Ax\|\left\|[A]x\right\|\le(m+M)\langle Ax,x\rangle\qquad\textit{for all } x\in H.$$

Proof. For $x\in H$ define $u_x:\mathbb{C}\mapsto H:\lambda\mapsto\lambda x$. Then, identifying $\mathbb{C}$ and $B(\mathbb{C})$ canonically,

$$u_x^*Au_x=\langle Ax,x\rangle\qquad\text{for } A\in B(H).$$

There is nothing to prove if $[A]x=0$, otherwise put $u=u_{x/\|[A]x\|}$. The first assertion follows from Corollary 2.1. The second assertion is a direct consequence of the the first. □

Remark 2.1 *The second assertion in Theorem 2.4 may be proved in one line:*

$$\begin{aligned}&(m+M)^2\langle Ax,x\rangle^2-4Mm\|Ax\|^2\|[A]x\|^2\\&=\left\{2mM\|[A]x\|^2-(m+M)\langle Ax,x\rangle\right\}^2\\&\qquad+4Mm\big\langle(M-A)(A-m)[A]x,[A]x\big\rangle\left\|[A]x\right\|^2\ge 0\end{aligned}$$

Generalizations of the Kantorovich inequality have made significant progress. The Mathematical Society was given a treat in the form of topics for the Kantorovich inequality for a while.

On the other hand, in pursuit of an even simpler proof, in such a flood of papers, Nakamura [237] instantly presents the following result in Proceedings of the Japan Academy. It was in 1960, just one year after the paper due to Greub and Rheinboldt was published. It is a simple visual proof by using the concavity of $f(t) = t^{-1}$.

Theorem 2.5 *For $0 < m < M$, the following inequality holds true:*

$$\int_m^M t d\mu(t) \cdot \int_m^M \frac{1}{t} d\mu(t) \leq \frac{(M+m)^2}{4Mm} \tag{2.19}$$

for any positive Stieltjes measure μ on $[m, M]$ with $\|\mu\| = 1$.

It is easy to see, by the Gelfand representation of the C*-algebra generated by A and the identity operator I, that Theorem 2.5 implies the Kantorovich inequality.

If Nakamura had the opportunity to read [156] in an English translation and if he asked the mathematical community for judgment on the inequality (2.19) and its overwhelmingly simple proof, then how would that turn out? In one possible outcome, mathematicians would mostly get the impression that it was very easy to prove that result and therefore the investigations related to the Kantorovich inequality would be brought to the end. For some reason, Nakamura's paper is overlooked in the mathematical world.
To the best of this author's knowledge, there is no evidence that anyone has ever cited Nakamura's paper. Instead, several improvements to proofs of the Kantorovich inequality have been independently developed in Europe.

The origin of the Kantorovich inequality might be the following case of finite sequences.

Theorem 2.6 *If the sequence $\{\gamma_i\}$ satisfies the conditions such that $m \leq \gamma_i \leq M$ for some scalars $0 < m \leq M$ and $i = 1, 2, \cdots, n$, then*

$$(\xi_1\gamma_1 + \cdots + \xi_n\gamma_n)(\xi_1\gamma_1^{-1} + \cdots + \xi_n\gamma_n^{-1}) \leq \frac{(M+m)^2}{4Mm} \tag{2.20}$$

holds for every $\xi_i \geq 0$ such that $\xi_1 + \cdots + \xi_n = 1$.

First of all, we present a direct proof due to Henrici [141]:

Proof of Theorem 2.6. We may assume that $m < M$. Determine p_i and q_i from the equations

$$\gamma_i = p_i M + q_i m \quad \text{and} \quad \gamma_i^{-1} = p_i M^{-1} + q_i m^{-1} \quad \text{for} \quad i = 1, \cdots, n.$$

An easy computation shows that $p_i, q_i \geq 0$, $i = 1, 2, \cdots, n$. Furthermore from

$$1 = (p_i M + q_i m)(p_i M^{-1} + q_i m^{-1}) = (p_i + q_i)^2 + p_i q_i \frac{(M-m)^2}{mM}$$

it follows that $p_i + q_i \le 1$. Setting $p = \sum \xi_i p_i$, $q = \sum \xi_i q_i$, we thus have $p + q = \sum \xi_i (p_i + q_i) \le \sum \xi_i = 1$. Hence using the arithmetic-geometric mean inequality,

$$\begin{aligned}
&(\xi_1\gamma_1 + \cdots + \xi_n\gamma_n)(\xi_1\gamma_1^{-1} + \cdots + \xi_n\gamma_n^{-1}) \\
&= (pM + qm)(pM^{-1} + qm^{-1}) = (p+q)^2 + pq\frac{(M-m)^2}{Mm} \\
&\le (p+q)^2\left[1 + \frac{(M-m)^2}{4Mm}\right] = (p+q)^2\frac{(M+m)^2}{4Mm} \le \frac{(M+m)^2}{4Mm}.
\end{aligned}$$

Equality is attained in (2.20) if and only if the following two conditions are simultaneously fulfilled (we assume here $\xi_i > 0, i = 1, 2, \cdots, n$ without loss of generalization):

(i) $p + q = 1$. This implies that $p_i + q_i = 1$ or $p_i q_i = 0$ for $i = 1, \cdots, n$. Thus, for equality every γ_i must equal either M or m.

(ii) $p + q = 4pq$. This implies that $p = q$ or, $\sum_{\gamma_i = m} \xi_i = \sum_{\gamma_i = M} \xi_i$.

Thus, the weights attached to m and M must be the same. □

In comparison with Kantorovich's proof, Henrici's one relies on an algebraic calculation. Inspired by Henrici, Rennie [255] gives the following improved proof with functions in 1963:

Let f be a measurable function on the probability space such that $0 < m \le f(x) \le M$. Integrating the inequality

$$\frac{(f(x) - m)(f(x) - M)}{f(x)} \le 0$$

gives

$$\int f(x)\mathrm{d}x + mM\int \frac{1}{f(x)}\mathrm{d}x \le m + M.$$

Put $u = mM\int \frac{1}{f(x)}\mathrm{d}x$, then we have

$$u\int f(x)\mathrm{d}x \le (m+M)u - u^2 = -\left(u - \frac{M+m}{2}\right)^2 + \frac{(M+m)^2}{4} \le \frac{(M+m)^2}{4},$$

which is the Kantorovich inequality:

$$\int \frac{1}{f(x)}\mathrm{d}x \int f(x)\mathrm{d}x \le \frac{(M+m)^2}{4mM}.$$

□

This is exactly a function version of the Kantorovich inequality due to Nakamura. Its emphatic brevity is surprising. Moreover, inspired by Rennie, Mond [209] gives the following improved proof with matrices in 1965:

Let A be a positive definite Hermitian matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. Since three factors in the LHS of below inequality commute, we have

$$(A - \lambda_n I)(A - \lambda_1 I)A^{-1} \le 0.$$

Therefore,

$$\langle Ax,x\rangle + \lambda_1\lambda_n\langle A^{-1}x,x\rangle \le \lambda_1 + \lambda_n$$

for every unit vector x. If we put $u = \lambda_1\lambda_n\langle A^{-1}x,x\rangle$, then

$$\lambda_1\lambda_n\langle A^{-1}x,x\rangle\langle Ax,x\rangle = u\langle Ax,x\rangle \le (\lambda_1+\lambda_n)u - u^2 \le \frac{(\lambda_1+\lambda_n)^2}{4},$$

which implies the Kantorovich inequality:

$$\langle A^{-1}x,x\rangle\langle Ax,x\rangle \le \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}.$$

The proof of Mond may be considered one of the generalized Kantorovich inequality. But, we present a somewhere different proof by using the arithmetic-geometric mean inequality in [164, 144, 158]:

Since A is positive and $0 < mI_H \le A \le MI_H$, it follows that $MI_H - A \ge 0$ and $A - mI_H \ge 0$. The commutativity of $MI_H - A$ and $A - mI_H$ implies $(MI_H - A)(m^{-1}I_H - A^{-1}) \ge 0$. Hence

$$(M+m)I_H \ge MmA^{-1} + A$$

and

$$\langle (M+m)x,x\rangle \ge Mm\langle A^{-1}x,x\rangle + \langle Ax,x\rangle$$

holds for every unit vector $x \in H$. By using the arithmetic-geometric mean inequality

$$M+m = \langle (M+m)x,x\rangle \ge Mm\langle A^{-1}x,x\rangle + \langle Ax,x\rangle \ge 2\sqrt{Mm\langle A^{-1}x,x\rangle\langle Ax,x\rangle}.$$

Squaring both sides, we obtain the desired inequality

$$\langle Ax,x\rangle\langle A^{-1}x,x\rangle \le \frac{(M+m)^2}{4Mm}.$$

Finally, we present an extremely simple idea due to Diaz and Metcalf [43]:

Lemma 2.3 *Let real numbers $a_k \ne 0$ and b_k $(k = 1,2,\cdots,n)$ satisfy*

$$m \le \frac{b_k}{a_k} \le M. \tag{2.21}$$

Then

$$\sum_{k=1}^{n} b_k^2 + mM\sum_{k=1}^{n} a_k^2 \le (M+m)\sum_{k=1}^{n} a_k b_k.$$

The equality holds if and only if in each of the n inequalities (2.21)*, at least one of the equality signs holds, i.e. either $b_k = ma_k$ or $b_k = Ma_k$ (where the equation may vary with k).*

Proof. It follows from the hypothesis (2.21) that

$$0 \le \left(\frac{b_k}{a_k} - m\right)\left(M - \frac{b_k}{a_k}\right) a_k^2.$$

Thus, summing from $k = 1$ to $k = n$,

$$\begin{aligned} 0 &\le \sum_{k=1}^{n} (b_k - ma_k)(Ma_k - b_k) \\ &= (M+m)\sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{n} b_k^2 - mM\sum_{k=1}^{n} a_k^2, \end{aligned} \tag{2.22}$$

which gives the desired result. Clearly, the equality holds in (2.22) if and only if each term of the summation is zero. □

By using Lemma 2.3, we have

$$\begin{aligned} 0 &\le \left(\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} - \left(mM\sum_{k=1}^{n} a_k^2\right)^{1/2}\right)^2 \\ &= \sum_{k=1}^{n} b_k^2 - 2\left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\left(mM\sum_{k=1}^{n} a_k^2\right)^{1/2} + mM\sum_{k=1}^{n} a_k^2 \\ &\le (m+M)\sum_{k=1}^{n} a_k b_k - 2\sqrt{mM}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \end{aligned}$$

and hence

$$4mM\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} a_k^2\right) \le (m+M)^2\left(\sum_{k=1}^{n} a_k b_k\right)^2$$

yields immediately the result of Pólya and Szegö (Theorem PS (2.2)).

Similarly, we have an operator version of Lemma 2.3:

Theorem 2.7 *Let A and B be self-adjoint operators such that $AB = BA$ and A^{-1} exists, and*

$$mI_H \le BA^{-1} \le MI_H \qquad \text{for some scalars } 0 < m \le M.$$

Then

$$B^2 + mMA^2 \le (m+M)AB. \tag{2.23}$$

The equality holds in (2.23) *if and only if* $(MI_H - BA^{-1})(BA^{-1} - mI_H) = 0$.

By using Theorem 2.7, we have

$$\begin{aligned} 0 &\le \left\{\langle Bx, Bx\rangle^{1/2} - mM\langle Ax, Ax\rangle^{1/2}\right\}^2 \\ &= \langle Bx, Bx\rangle - 2\sqrt{mM}\langle Bx, Bx\rangle^{1/2}\langle Ax, Ax\rangle^{1/2} + mM\langle Ax, Ax\rangle \\ &\le (m+M)\langle ABx, x\rangle - 2\sqrt{mM}\langle Bx, Bx\rangle^{1/2}\langle Ax, Ax\rangle^{1/2} \end{aligned}$$

and hence

$$4mM\langle Bx,Bx\rangle\langle Ax,Ax\rangle \leq (m+M)^2\langle ABx,x\rangle^2$$

yields immediately results of Greub and Rheinboldt (Theorem 2.1).

Comparing with the proofs of Kantorovich and Greub and Rheinboldt, only algebraic calculation seems to belong to a different age. However, when we can prove it plainly and simply, devising a new proof stops being an object of interest for mathematicians.

2.3 The Mond-Pečarić method

In this section, we present the principle of the Mond-Pečarić method for convex functions.

Mond and Pečarić rephrased the Kantorovich inequality as follows: The Kantorovich inequality says that if A is a positive operator such that $0 < mI_H \leq A \leq MI_H$, then

$$\langle Ax,x\rangle\langle A^{-1}x,x\rangle \leq \frac{(M+m)^2}{4Mm} \tag{2.24}$$

for every unit vector $x \in H$. Divideing both sides by $\langle Ax,x\rangle$, we get

$$\langle A^{-1}x,x\rangle \leq \frac{(M+m)^2}{4Mm}\langle Ax,x\rangle^{-1}. \tag{2.25}$$

Also, since $1 \leq \langle Ax,x\rangle\langle A^{-1}x,x)$, we may extend (2.25) into the following inequality:

$$\langle Ax,x\rangle^{-1} \leq \langle A^{-1}x,x\rangle \leq \frac{(M+m)^2}{4Mm}\langle Ax,x\rangle^{-1}. \tag{2.26}$$

The first inequality of (2.26) is a special case of Jensen's inequality. In fact, if we put $f(t) = t^{-1}$, then

$$\left(\frac{a_1+\cdots+a_n}{n}\right)^{-1} \leq \frac{a_1^{-1}+\cdots+a_n^{-1}}{n}$$

for all positive real numbers $a_1,\cdots,a_n$. Moreover, if $f(t)$ is a convex function on an interval $[m,M]$, then

$$f\left(\sum_{i=1}^n t_i x_i\right) \leq \sum_{i=1}^n t_i f(x_i)$$

for every $x_1,\cdots,x_n \in [m,M]$ and every positive real number $t_1,\cdots,t_n$ with $\sum_{i=1}^n t_i = 1$. This inequality is called the classical Jensen's inequality. Moreover, an operator version of the classical Jensen's inequality holds:

Theorem 2.8 *Let A be a self-adjoint operator on H such that $mI_H \leq A \leq MI_H$ for some scalars $m \leq M$ and f a real valued continuous convex function on $[m,M]$. Then*

$$f(\langle Ax,x\rangle) \leq \langle f(A)x,x\rangle$$

holds for every unit vector $x \in H$.

Proof. Refer to [124, Theorem 1.2] for the proof. □

From this point of view, $\langle Ax,x\rangle^{-1} \leq \langle A^{-1}x,x\rangle$ is considered as one form of Jensen's inequality. Namely, Mond and Pečarić noticed that

the Kantorovich inequality is the converse inequality of the so called Jensen's one for the function $f(t) = 1/t$.

Jensen's inequality is one of the most important inequalities in the functional analysis. Many generalizations are developed and many significant results are obtained by using Jensen's inequalitiy.

Here, let us consider a generalization of the Kantorovich inequality. Jensen's inequality for $f(t) = t^3$ yields

$$\langle Ax,x\rangle^3 \leq \langle A^3x,x\rangle \qquad \text{for every unit vector } x \in H. \tag{2.27}$$

What is a converse of (2.27)? Unfortunately, it seems to be difficult to apply the same method as in the proof of the Kantorovich inequality. We need a new way of thinking. We recall Nakamura's article [237]. It was published too early, as it was ahead of its time and later on hardly anyone looked back at that paper. Thirty years later ideas similar to his had appeared in Eastern Europe. By then Nakamura had forgotten all about his principle, but it had taken root in Eastern Europe and would grow in time.

Thus, we shall recall the proof due to Nakamura: Let μ be a normalized positive Stieltjes measure on $[m,M]$. Let $y = g(t)$ a straight line joining the points $(m,1/m)$ and $(M,1/M)$. Since $1/t \leq g(t)$, we have

$$\int_m^M \frac{1}{t} d\mu(t) \leq \int_m^M g(t) d\mu(t) = \frac{M^{-1}+m^{-1}}{2}.$$

Multiply $\int_m^M t d\mu(t) = \frac{M+m}{2}$ to both sides,

$$\int_m^M t d\mu(t) \int_m^M \frac{1}{t} d\mu(t) \leq \frac{M+m}{2} \cdot \frac{M^{-1}+m^{-1}}{2} = \frac{(M+m)^2}{4Mm}.$$

Applying it to a positive operator A with $\|A\| = M$ and $\|A^{-1}\|^{-1} = m$, we have just the Kantorovich inequality

$$\langle Ax,x\rangle\langle A^{-1}x,x\rangle \leq \frac{(M+m)^2}{4Mm}$$

for every unit vector $x \in H$. We remark that the Kantorovich constant equals the arithmetic mean of m and M divided by the harmonic one:

$$\frac{(M+m)^2}{4Mm} = \frac{\frac{M+m}{2}}{\left(\frac{M^{-1}+m^{-1}}{2}\right)^{-1}}.$$

Namely, we know that Nakamura's proof is actually the origin of the so called the Mond-Pečarić method for convex functions by which the converses of Jensen's inequality are induced. Moreover, Ky Fan [48] proceeded with a generalization of the Kantorovich inequality for $f(t) = t^p$ with $p \in \mathbb{Z}$. Here, we shall present the principle of the Mond-Pečarić method for convex functions:

Theorem 2.9 *Let A be a self-adjoint operator on a Hilbert space H such that $mI_H \leq A \leq MI_H$ for some scalars $m < M$. If f is a convex function on $[m,M]$ such that $f > 0$ on $[m,M]$, then*

$$\langle f(A)x,x\rangle \leq K(m,M,f)f(\langle Ax,x\rangle)$$

for every unit vector $x \in H$, where

$$K(m,M,f) = \max\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right) : m \leq t \leq M\right\}.$$

Proof. Since $f(t)$ is convex on $[m,M]$, we have

$$f(t) \leq \frac{f(M)-f(m)}{M-m}(t-m)+f(m) \qquad \text{for all } t \in [m,M].$$

Using the operator calculus, it follows that

$$f(A) \leq \frac{f(M)-f(m)}{M-m}(A-m)+f(m)I_H$$

and hence

$$\langle f(A)x,x\rangle \leq \frac{f(M)-f(m)}{M-m}(\langle Ax,x\rangle - m)+f(m)$$

for every unit vector $x \in H$. Divide both sides by $f(\langle Ax,x\rangle)$ (>0), and we get

$$\begin{aligned}\frac{\langle f(A)x,x\rangle}{f(\langle Ax,x\rangle)} &\leq \frac{\frac{f(M)-f(m)}{M-m}(\langle Ax,x\rangle - m)+f(m)}{f(\langle Ax,x\rangle)} \\ &\leq \max\left\{\frac{1}{f(t)}\left(\frac{f(M)-f(m)}{M-m}(t-m)+f(m)\right) : m \leq t \leq M\right\},\end{aligned}$$

since $m \leq \langle Ax,x\rangle \leq M$. Therefore, we have the desired inequality. □

Theorem 2.10 *Let A be a self-adjoint operator on a Hilbert space H such that $mI_H \leq A \leq MI_H$ for some scalars $m < M$. If f is a concave function on $[m,M]$ such that $f > 0$ on $[m,M]$, then*

$$\tilde{K}(m,M,f)f(\langle Ax,x\rangle) \leq \langle f(A)x,x\rangle \leq f(\langle Ax,x\rangle)$$