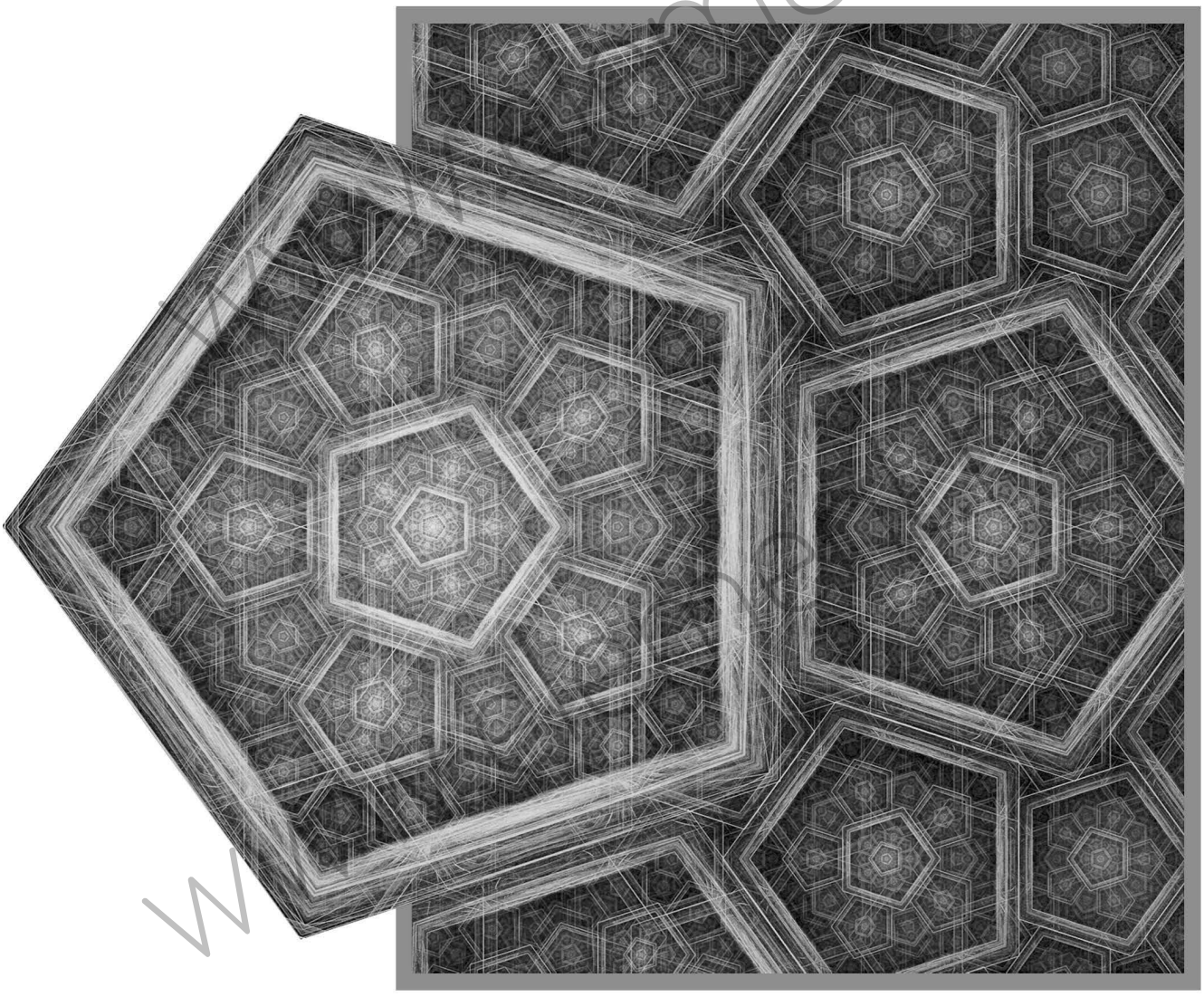
1 Skup kompleksnih brojeva



1.1. Kompleksni broj

Zadatak 1. Odredi realni i imaginarni dio svakog od kompleksnih brojeva:

1) $z = 5 + 2i$; **2)** $z = 1 - 3i$;

3) $z = -\frac{1}{2}i$; **4)** $z = \sqrt{2}$;

5) $z = \frac{2-3i}{3}$; **6)** $z = 1 - \sqrt{2} + i\sqrt{3}$;

7) $z = 0$; **8)** $z = (1 - \sqrt{2})i$.

Rješenje. **1)** $\operatorname{Re} z = 5$, $\operatorname{Im} z = 2$; **2)** $\operatorname{Re} z = 1$, $\operatorname{Im} z = -3$;

3) $\operatorname{Re} z = 0$, $\operatorname{Im} z = -\frac{1}{2}$; **4)** $\operatorname{Re} z = \sqrt{2}$, $\operatorname{Im} z = 0$;

5) $\operatorname{Re} z = \frac{2}{3}$, $\operatorname{Im} z = -1$; **6)** $\operatorname{Re} z = 1 - \sqrt{2}$, $\operatorname{Im} z = \sqrt{3}$;

7) $\operatorname{Re} z = 0$, $\operatorname{Im} z = 0$; **8)** $\operatorname{Re} z = 0$, $\operatorname{Im} z = 1 - \sqrt{2}$.

Zadatak 2. Odredi realne brojeve a i b iz jednakosti:

1) $a + 4i = -2 + bi$; **2)** $a + i = 1 + bi$;

3) $2a - b + 3i = a + 2b - 3i$; **4)** $1 - (a - b)i = 1 + (a + b)i$;

5) $a - b + 5i = 1 + (a + b)i$; **6)** $-1 + (2a + b)i = 1 - (a - 2b)i$.

Rješenje. **1)** $a = -2$, $b = 4$; **2)** $a = 1$, $b = 1$; **3)** nema rješenja;
4) $a = 0$, b može biti bilo koji realni broj; **5)** $a = 3$, $b = 2$.
6) nema rješenja.

Zadatak 3. Odredi realne brojeve x i y iz jednakosti:

1) $x + (y - 1)i = -1 + 3i$;
2) $2x + y - yi = 1 + i$;
3) $x - y + (x + y)i = 2 + 4i$;
4) $x - 2y + (2x - y)i = 3i$;
5) $2x - 3y + (x - y)i = -1$;
6) $2x - 3y + (x + y)i = x + 2y + (3x + 1)i$.

Rješenje. **1)** Kompleksni brojevi s lijeve i desne strane jednakosti jednaki su ako je $x = -1$ i $y - 1 = 3$, odnosno $x = -1$, $y = 4$;
2) Mora biti $2x + y = 1$ i $-y = 1$, odnosno $x = 1$, $y = -1$;
3) Mora biti $x - y = 2$ i $x + y = 4$, odnosno $x = 3$, $y = 1$;
4) Rješavanjem sustava $x - 2y = 0$, $2x - y = 3$ dobivamo $x = 2$, $y = 1$;
5) Rješavanjem sustava $2x + 3y = -1$, $x - y = 0$ dobivamo $x = y = 1$.
6) $x = -\frac{5}{9}$, $y = -\frac{1}{9}$.

Zadatak 4. Gdje je greška u računu:

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1}$$
$$= i\cdot i = -1?$$

Rješenje. Svojstvo drugog korijena umnoška, tj. $\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$ vrijedi samo za pozitivne brojeve a i b. Stoga ne vrijedi jednakost $\sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1}$.

1.2. Zbrajanje i množenje kompleksnih brojeva

Zadatak 1. Izračunaj $z+w$, $z-w$ i $z\cdot w$ ako je:

1) $z = -\frac{1}{2}+i,\ w = 1-\frac{1}{3}i$; **2)** $z = -2+3i,\ w = 2+i$;

3) $z = \frac{3}{4}-\frac{2}{3}i,\ w = \frac{3}{4}+\frac{1}{3}i$.

Rješenje. **1)** $z+w = -\frac{1}{2}+i+1-\frac{1}{3}i = \frac{1}{2}+\frac{2}{3}i$, $z-w = -\frac{1}{2}+i-\left(1-\frac{1}{3}i\right) = -\frac{3}{2}+\frac{4}{3}i$,
$z\cdot w = \left(-\frac{1}{2}+i\right)\left(1-\frac{1}{3}i\right) = -\frac{1}{2}+\frac{1}{6}i+i-\frac{1}{3}i^2 = -\frac{1}{2}+\frac{7}{6}i+\frac{1}{3} = -\frac{1}{6}+\frac{7}{6}i$;
2) $z+w = -2+3i+2+i = 4i$, $z-w = -2+3i-(2+i) = -4+2i$,
$z\cdot w = (-2+3i)\cdot(2+i) = -4-2i+6i+3i^2 = -4+4i-3 = -7+4i$;
3) $z+w = \frac{3}{4}-\frac{2}{3}i+\frac{3}{4}+\frac{1}{3}i = \frac{3}{2}-\frac{1}{3}i$, $z-w = \frac{3}{4}-\frac{2}{3}i-\left(\frac{3}{4}+\frac{1}{3}i\right) = -i$,
$z\cdot w = \left(\frac{3}{4}-\frac{2}{3}i\right)\cdot\left(\frac{3}{4}+\frac{1}{3}i\right) = \frac{9}{16}-\frac{1}{4}i-\frac{1}{2}i-\frac{2}{9}i^2 = \frac{9}{16}-\frac{3}{4}i+\frac{2}{9} = \frac{113}{144}-\frac{3}{4}i$.

Zadatak 2. Izračunaj $z+w$, $z-w$, $z\cdot w$, z^2 i w^2 ako je:

1) $z = 1-2i,\ w = 3-i$;
2) $z = -3+5i,\ w = 4-7i$;
3) $z = 11-5i,\ w = -7-i$.

Rješenje. **1)** $z+w = 1-2i+3-i = 4-3i$, $z-w = 1-2i-(3-i) = -2-i$,
$z\cdot w = (1-2i)\cdot(3-i) = 3-i-6i+2i^2 = 3-7i-2 = 1-7i$,
$z^2 = (1-2i)^2 = 1-4i+4i^2 = 1-4i-4 = -3-4i$, $w^2 = (3-i)^2$
$= 9-6i+i^2 = 9-6i-1 = 8-6i$;
2) $z+w = -3+5i+4-7i = 1-2i$, $z-w = -3+5i-(4-7i) = -7+12i$,
$z\cdot w = (-3+5i)(4-7i) = -12+21i+20i-35i^2 = -12+41i+35 = 23+41i$,
$z^2 = (-3+5i)^2 = 9-30i+25i^2 = 9-30i-25 = -16-30i$,
$w^2 = (4-7i)^2 = 16-56i+49i^2 = 16-56i-49 = -33-56i$;
3) $z+w = 11-5i+(-7-i) = 4-6i$, $z-w = 11-5i-(-7-i) = 18-4i$,
$z\cdot w = (11-5i)(-7-i) = -77-11i+35i+5i^2 = -77+24i-5 = -82+24i$,
$z^2 = (11-5i)^2 = 121-110i+25i^2 = 121-110i-25 = 96-110i$,
$w^2 = (-7-i)^2 = 49+14i+i^2 = 49+14i-1 = 48+14i$.

Zadatak 3. **1)** $(1+i)^2$; **2)** $(1-2i)^2$; **3)** $(2-i)^2$; **4)** $(1+2i)^3$;
5) $(3+2i)^3$; **6)** $(i+2)^3$; **7)** $(1-i)^4$; **8)** $(2+i)^4$.

Rješenje. **1)** $(1+i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$;
2) $(1-2i)^2 = 1-4i+4i^2 = 1-4i-4 = -3-4i$;
3) $(2-i)^2 = 4-4i+i^2 = 4-4i-1 = 3-4i$;
4) $(1+2i)^3 = 1+6i+12i^2+8i^3 = 1+6i-12-8i = -11-2i$;
5) $(3+2i)^3 = 27+54i+36i^2+8i^3 = 27+54i-36-8i = -9+46i$;
6) $(i+2)^3 = i^3+6i^2+12i+8 = -i-6+12i+8 = 2+11i$;
7) $(1-i)^4 = ((1-i)^2)^2 = (1-2i+i^2)^2 = (-2i)^2 = -4$;
8) $(2+i)^4 = ((2+i)^2)^2 = (4+4i-1)^2 = (3+4i)^2 = 9+24i-16 = -7+24i$.

Zadatak 4. **1)** $(1-i\sqrt{2})(\sqrt{2}-i)$;
2) $(\sqrt{3}-i)(1+i\sqrt{3})$;
3) $(\sqrt{2}-i)(\sqrt{3}+2i)-(\sqrt{3}+i)(\sqrt{2}-i)$.

Rješenje. **1)** $(1-i\sqrt{2})(\sqrt{2}-i) = \sqrt{2}-i-2i+\sqrt{2}i^2 = \sqrt{2}-3i-\sqrt{2} = -3i$;
2) $(\sqrt{3}-i)(1+i\sqrt{3}) = \sqrt{3}+3i-i-\sqrt{3}i^2 = \sqrt{3}+2i+\sqrt{3} = 2\sqrt{3}+2i$;
3) $(\sqrt{2}-i)(\sqrt{3}+2i)-(\sqrt{3}+i)(\sqrt{2}-i)$
$= \sqrt{6}+2\sqrt{2}i-\sqrt{3}i-2i^2-(\sqrt{6}-\sqrt{3}i+\sqrt{2}i-i^2)$
$= \sqrt{6}+2\sqrt{2}i-\sqrt{3}i+2-\sqrt{6}+\sqrt{3}i-\sqrt{2}i-1 = 1+\sqrt{2}i$.

Zadatak 5. **1)** $(1-i)(2-i)(3-i)$;
2) $(1+i)(1+2i)(1+3i)$;
3) $\left(\frac{1}{2}-i\right)(1+2i)\left(1-\frac{1}{2}i\right)(2+i)$.

Rješenje. **1)** $(1-i)(2-i)(3-i) = (2-3i-1)(3-i) = (1-3i)(3-i)$
$= 3-10i+3 = -10i$;
2) $(1+i)(1+2i)(1+3i) = (1+3i-2)(1+3i) = (3i-1)(3i+1)$
$= 9i^2-1 = -9-1 = -10$;
3) $\left(\frac{1}{2}-i\right)(1+2i)\left(1-\frac{1}{2}i\right)(2+i) = \left(\frac{1}{2}+i-i+2\right)\left(2+i-i+\frac{1}{2}\right) = \frac{5}{2}\cdot\frac{5}{2} = \frac{25}{4}$.

Zadatak 6. **1)** $(1-i)^2\cdot(2-i)^2\cdot(3-i)^2$; **2)** $(1-i)^2\cdot(1-2i)^2\cdot(1-3i)^2$;
3) $(1+i)^3\cdot(1+2i)^3\cdot(1+3i)^3$; **4)** $(1+2i)^3\cdot(2-i)^3\cdot(1+3i)^3\cdot(3-i)^3$.

Rješenje. **1)** $(1-i)^2\cdot(2-i)^2\cdot(3-i)^2 = [(1-i)(2-i)(3-i)]^2 = [(1-3i)(3-i)]^2$
$= (-10i)^2 = 100i^2 = -100$;
2) $(1-i)^2\cdot(1-2i)^2\cdot(1-3i)^2 = [(1-i)(1-2i)(1-3i)]^2 = [(-1-3i)(1-3i)]^2$
$= (9i^2-1)^2 = (-10)^2 = 100$;
3) $(1+i)^3\cdot(1+2i)^3\cdot(1+3i)^3 = [(1+i)(1+2i)(1+3i)]^3 = [(-1+3i)(1+3i)]^3$
$= [9i^2-1]^3 = (-10)^3 = -1000$;
4) $(1+2i)^3\cdot(2-i)^3\cdot(1+3i)^3\cdot(3-i)^3 = [(1+2i)(2-i)(1+3i)(3-i)]^3$
$= [(4+3i)(6+8i)]^3 = (24+32i+18i-24)^3 = (50i)^3 = -125\,000i$.

Zadatak 7.

1) $(1-\sqrt{2}+i)(1+\sqrt{2}-i)$;

2) $\Big(1-(\sqrt{2}-\sqrt{3})i\Big)\Big(1+(\sqrt{2}+\sqrt{3})i\Big)$;

3) $(1-\sqrt{2}+i\sqrt{3})\cdot(1+\sqrt{2}-i\sqrt{3})\cdot(1-\sqrt{2}-i\sqrt{3})\cdot(1+\sqrt{2}+i\sqrt{3})$.

Rješenje.

1) $(1-\sqrt{2}+i)(1+\sqrt{2}-i)=(1-(\sqrt{2}-i))(1+(\sqrt{2}-i))=1^2-(\sqrt{2}-i)^2$
$=1-(2-2\sqrt{2}i+i^2)=1-2+2i\sqrt{2}+1=2\sqrt{2}i$;

2) $\Big(1-(\sqrt{2}-\sqrt{3})i\Big)\Big(1+(\sqrt{2}+\sqrt{3})i\Big)$
$=1+(\sqrt{2}+\sqrt{3})i-(\sqrt{2}-\sqrt{3})i-(\sqrt{2}+\sqrt{3})i\cdot(\sqrt{2}-\sqrt{3})i$
$=1+2i\sqrt{3}+(\sqrt{2}^2-\sqrt{3}^2)=1+2i\sqrt{3}-1=2i\sqrt{3}$;

3) $(1-\sqrt{2}+i\sqrt{3})\cdot(1+\sqrt{2}-i\sqrt{3})\cdot(1-\sqrt{2}-i\sqrt{3})\cdot(1+\sqrt{2}+i\sqrt{3})$
$=(1-\sqrt{2}+i\sqrt{3})(1+\sqrt{2}+i\sqrt{3})\cdot(1+\sqrt{2}-i\sqrt{3})(1-\sqrt{2}-i\sqrt{3})$
$=((1+i\sqrt{3})^2-(\sqrt{2})^2)((1-i\sqrt{3})^2-(\sqrt{2})^2)$
$=(1+2i\sqrt{3}-3-2)(1-2i\sqrt{3}-3-2)=(-4+2i\sqrt{3})(-4-2i\sqrt{3})$
$=16-(2i\sqrt{3})^2=16+12=28$.

Zadatak 8. Izračunaj vrijednost brojevnog izraza

$$z^2-z\cdot w+w^2$$

ako je:

1) $z=1-2i,\ w=3+i$;

2) $z=\sqrt{2}-i,\ w=\sqrt{2}+i$;

3) $z=1+\sqrt{2}-\sqrt{3}\,i,\ w=1-\sqrt{2}+\sqrt{3}i$.

Rješenje.

1) $(1-2i)^2-(1-2i)(3+i)+(3+i)^2=1-4i-4-3-i+6i-2+9+6i-1=7i$;

2) $(\sqrt{2}-i)^2-(\sqrt{2}-i)\cdot(\sqrt{2}+i)+(\sqrt{2}+i)^2$
$=2-2i\sqrt{2}-1-2-1+2+2i\sqrt{2}-1=-1$;

3) $(1+\sqrt{2}-\sqrt{3}i)^2-(1+\sqrt{2}-\sqrt{3}i)(1-\sqrt{2}+\sqrt{3}i)+(1-\sqrt{2}+\sqrt{3}i)^2$
$=1+2\sqrt{2}+2-2\sqrt{3}i-2\sqrt{6}i-3-(1^2-(\sqrt{2}-\sqrt{3}i)^2)+1-2\sqrt{2}+2$
$+2\sqrt{3}i-2\sqrt{6}i-3=-2\sqrt{6}i-(1-2+2\sqrt{6}i+3)-2\sqrt{6}i$
$=-4\sqrt{6}i-2-2\sqrt{6}i=-2-6\sqrt{6}i$.

Zadatak 9. Izračunaj vrijednost izraza za zadanu vrijednost kompleksnog broja:

1) z^3-z^2+2z, za $z_1=1+i,\ z_2=1-i$;

2) z^3+3z^2-z+1, za $z_1=2+i,\ z_2=2-i$;

3) z^4-z^2+2, za $z_1=1+i,\ z_2=1-i$.

Rješenje.

1) $z^3-z^2+2z=(1+i)^3-(1+i)^2+2(1+i)$
$=1+3i-3-i-1-2i+1+2+2i=2i$,
$z^3-z^2+2z=(1-i)^3-(1-i)^2+2(1-i)=1-3i-3+i-1+2i+1+2-2i=-2i$;

2) $z^3+3z^2-z+1=(2+i)^3+3(2+i)^2-(2+i)+1$
$=8+12i-6-i+12+12i-3-2-i+1=10+22i$,
$z^3+3z^2-z+1=(2-i)^3+3(2-i)^2-(2-i)+1$
$=8-12i-6+i+12-12i-3-2+i+1=10-22i$;