Prva kontrolna zadaća

2. 12. 1996.

1. Zadana je funkcija

$$S(z) = \frac{i}{1-z}.$$

a) Funkcijom $S(z)$ preslikati područje $D = \{z \in \mathbf{C} \mid |z| > 1,\ 0 < \mathrm{Re}\,(z) < 1,\ 0 < \mathrm{Im}\,(z) < 1\}$. Skicirati $S(D)$.

b) Funkcijom $f(z) = z + S(z)$ preslikati dužinu $\overline{AB}$, gdje je $A = \left(1, -\frac{1}{2}\right)$, $B = (1, 1)$. Skicirati $f(\overline{AB})$.

2. Izračunati integral

$$\int_\Gamma \frac{dz}{z^3 + i}$$

gdje je Γ zatvorena pozitivno orijentirana krivulja dana s $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 = \{z \in \mathbf{C} \mid \mathrm{Im}\,(z) = -\mathrm{Re}\,(z),\ -\sqrt{2} \leq \mathrm{Re}\,(z) \leq \sqrt{2}\}$, $\Gamma_2 = \{z \in \mathbf{C} \mid |z| = 2,\ \mathrm{Re}\,(z) \leq -\mathrm{Im}\,(z)\}$.

3. Razviti u Laurentov red funkciju

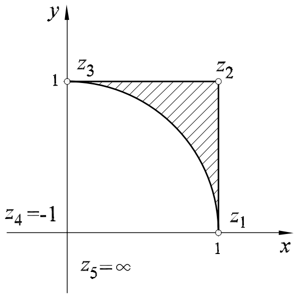
$$f(z) = \mathrm{arc\,tg}\,\frac{z-1}{4} + \frac{1}{(z+i)(z-1)}$$

oko točke $z_0 = 1$ u području D koje sadrži točku $z_1 = 0$. Odrediti područje konvergencije D.

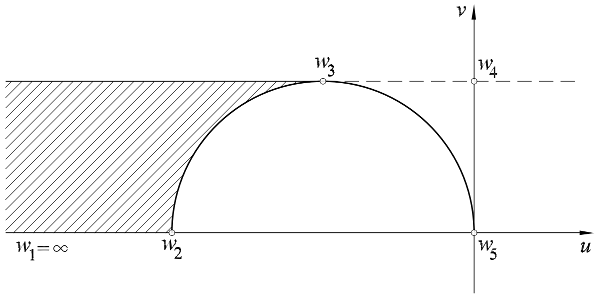
Rješenja

02. 12. 1996.

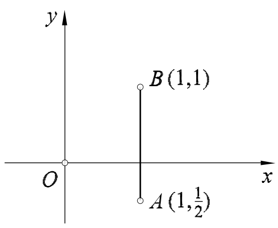
1. a) S obzirom da je zadano preslikavanje Möbiusova transformacija, izborom i preslikavanjem sljedećih 5 točaka možemo riješiti zadani problem:



$$
\begin{aligned}
z_1 &= 1 &&\to w_1 = \infty\\
z_2 &= 1+i &&\to w_2 = -1\\
z_3 &= i &&\to w_3 = -\frac{1}{2}+\frac{i}{2}\\
z_4 &= -1 &&\to w_4 = \frac{i}{2}\\
z_5 &= \infty &&\to w_5 = 0
\end{aligned}
$$



b)



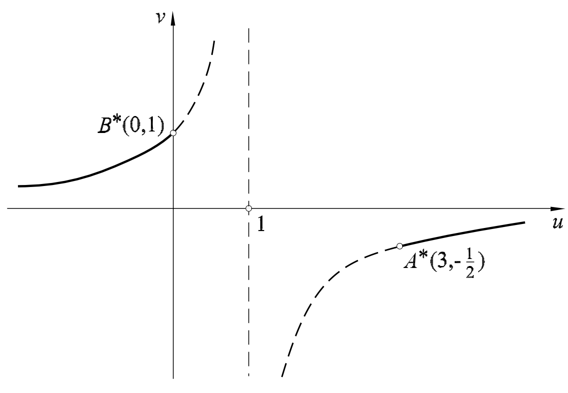
Ovdje nije moguće preslikavati po točkama. Zato uzmemo parametrizaciju zadane dužine

$$\overline{AB}\ldots z = 1+iy,\ y\in\left[-\frac{1}{2},1\right],$$

te je uvrstimo u funkciju f i dobijemo parametarski oblik krivulje koja predstavlja sliku dužine $\overline{AB}$.

$$f = 1+iy+\frac{i}{1-(1+iy)} = \underbrace{1-\frac{1}{y}}_{u}+i\underbrace{y}_{v}$$

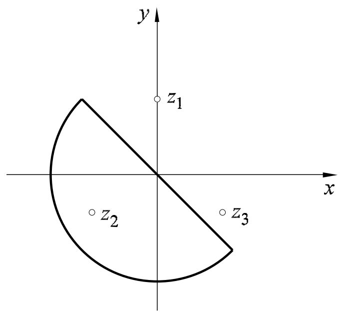
$$\Longrightarrow\quad v=\frac{1}{1-u},\quad v=y\in\left[-\frac{1}{2},1\right].$$



2. Jedina nultočka nazivnika koja se nalazi u području unutar Γ je

$$z_2 = -\frac{\sqrt{3}}{2}-\frac{1}{2}i$$

i to je pol prvoga reda. Zadatak se može dalje rješavati pomoću Cauchyeve integralne formule ili pomoću Teorema o ostatcima, kako ćemo mi dalje raditi.



$$
\begin{aligned}
I &= 2\pi i\cdot\ \mathrm{Res}\,(f,z_2) = 2\pi i\cdot\frac{1}{3z^2}\Big|_{z=z_2}\\
&= \frac{2\pi i}{3}\cdot\frac{1}{\frac{1}{2}+i\frac{\sqrt{3}}{2}} = \frac{4\pi i}{3}\cdot\frac{1-i\sqrt{3}}{4} = \frac{\pi}{3}(\sqrt{3}+i)
\end{aligned}
$$

3.

$$
\begin{aligned}
\mathrm{arc\,tg}\, z &= \int\frac{dz}{1+z^2} = \int\left(\sum_{n=0}^{\infty}(-1)^n z^{2n}\right)dz\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{2n+1}+C,\qquad |z|<1.
\end{aligned}
$$

Iz uvjeta $\mathrm{arc\,tg}\,0 = 0$ odmah dobivamo da je $C=0$. Zato je

$$\mathrm{arc\,tg}\frac{z-1}{4} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(z-1)^{2n+1}}{(2n+1)\cdot 4^{2n+1}},\quad |z-1|<4.$$

Drugi pribrojnik zadane funkcije razvijamo standardnom tehnikom u Laurentov razvoj, uzimajući u obzir $|z_1-z_0|<|-i-z_0|$.

$$
\begin{aligned}
\frac{1}{(z-1)(z+i)} &= \frac{1}{z-1}\cdot\frac{1}{z-1+i+1}\\
&= \frac{1}{z-1}\cdot\frac{1}{(i+1)(1+\frac{z-1}{i+1})}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\cdot(z-1)^{n-1}}{(i+1)^{n+1}}
\end{aligned}
$$

$$0<|z-1|<|1+i|=\sqrt{2}$$

$$
\begin{aligned}
\Longrightarrow\quad f(z) &= \frac{1}{(1+i)(z-1)}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z-1)^n}{(1+i)^{n+2}}\\
&+\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^{2n+1}}{(2n+1)\cdot 4^{2n+1}}.
\end{aligned}
$$

Područje konvergencije je presjek područja konvergencije istaknutih kod svakog od gore razvijenih pribrojnika, dakle

$$D=\{z\mid 0<|z-1|<\sqrt{2}\}.$$

Druga kontrolna zadaća

27. 1. 1997.

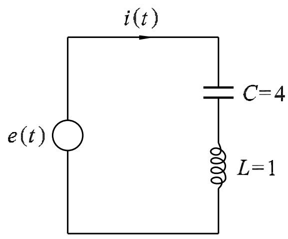
1. Prelaskom na integraciju u kompleksnom području izračunati integral

$$\int_{-\infty}^{\infty} \frac{\sin 3x}{x^2 - x}\, dx.$$

2. Funkciju $f(x) = \operatorname{sgn}(\cos \pi x)$ razviti u Fourierov red. Pomoću Parsevalove jednakosti izračunati red

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

3. Pomoću Laplaceove transformacije odrediti struju $i(t)$ strujnog kruga sa slike uz priključeni napon $e(t) = e^{-2t} \cdot S(t-1)$.



Rješenja

27. 1. 1997.

1. $I = \int\limits_{-\infty}^{\infty} \frac{\sin 3x}{x^2 - x}\,dx = \operatorname{Im} \int\limits_{-\infty}^{\infty} \frac{e^{3ix}}{x^2 - x}\,dx = \operatorname{Im} I_1.$

Ako provjerimo uvjet iz Jordanove leme, vrijedilo bi dalje

$$I_1 = \pi i [\operatorname{Res}(F, 0) + \operatorname{Res}(F, 1)].$$

Provjerimo uvjet iz Jordanove leme, te uzmimo z sa središnje polukružnice iz gornje poluravnine, $z = Re^{i\varphi}$, $R > 1$ i računajmo:

$$\begin{aligned}|f(z)| &= \frac{1}{|z^2 - z|} = \frac{1}{|z|\,|z-1|} \\ &\leq \frac{1}{|z|(|z|-1)} = \frac{1}{R(R-1)} \to 0, \text{ kad } R \to \infty.\end{aligned}$$

Izračunamo reziduume u polovima prvog reda

$$\begin{aligned}\operatorname{Res}(F, 0) &= \left.\frac{e^{3iz}}{2z-1}\right|_{z=0} = -1, \\ \operatorname{Res}(F, 1) &= \left.\frac{e^{3iz}}{2z-1}\right|_{z=1} = \cos 3 + i \sin 3.\end{aligned}$$

Stoga slijedi

$$I_1 = \pi i(-1 + \cos 3 + i \sin 3)$$

pa je traženi integral

$$I = \pi(\cos 3 - 1).$$

2. Očito je f periodička s periodom $T = 2$, a također je f parna. Zato ćemo rabiti formule za razvoj parnih funkcija, gdje je $L = 1$, te su koeficijenti $b_n = 0$. Uočimo eksplicitno: $f(x) = 1$, za $x \in [0, 1/2\rangle$, te $f(x) = -1$, za $x \in \langle 1/2, 1]$. Računamo:

$$\begin{aligned}a_0 &= \frac{2}{1}\left[\int\limits_0^{1/2} 1\,dx + \int\limits_{1/2}^{1} (-1)\,dx\right] \\ &= 2\left[\frac{1}{2} - \frac{1}{2}\right] = 0 \\ a_n &= \frac{2}{1}\left[\int\limits_0^{1/2} \cos(n\pi x)\,dx + \int\limits_{1/2}^{1} (-1)\cos(n\pi x)\,dx\right] \\ &= 2\left[\left.\frac{\sin n\pi x}{n\pi}\right|_{x=0}^{1/2} - \left.\frac{\sin n\pi x}{n\pi}\right|_{1/2}^{1}\right] \\ &= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right).\end{aligned}$$

Točnije, uočavamo

$$\Longrightarrow a_{2k} = 0, \quad a_{2k+1} = \frac{4}{(2k+1)\pi} \cdot (-1)^k$$

pa je traženi trigonometrijski Fourierov red

$$f(x) = \sum_{k=0}^{\infty} \frac{4}{\pi} \cdot \frac{(-1)^k}{2k+1} \cos[(2k+1)\pi x].$$

Parsevalova jednakost neposredno daje:

$$\sum_{k=0}^{\infty} \left(\frac{4}{\pi(2k+1)}\right)^2 = \frac{2}{2} \int\limits_{-1}^{1} 1^2\,dx$$

$$\Longrightarrow \frac{16}{\pi^2} \cdot s = 2 \Longrightarrow s = \frac{\pi^2}{8},$$

gdje smo sa s označili traženi zbroj.

3. Transformiramo zadanu naponsku funkciju kako bismo je primjenom Teorema o pomaku i Teorema o prigušenju lako preslikali u donje područje.

$$\begin{aligned}e(t) &= e^{-2t} \cdot S(t-1) = e^{-2} \cdot e^{-2(t-1)} \cdot S(t-1) \\ \bigcirc\!\!-\!\circ\; E(p) &= e^{-2} \cdot \frac{1}{p+2} \cdot e^{-p}.\end{aligned}$$

Ukupni otpor u donjem području je

$$Z(p) = \frac{1}{4p} + p = \frac{4p^2 + 1}{4p}$$

pa je zato struja u donjem području jednaka

$$\begin{aligned}I(p) &= \frac{E(p)}{Z(p)} = \frac{4p}{4p^2+1} \cdot e^{-2} \cdot \frac{1}{p+2} \cdot e^{-p} \\ &= e^{-2} \cdot \frac{p}{(p^2 + \frac{1}{4})(p+2)} \cdot e^{-p} \\ &= \text{rastav u parcijalne razlomke} \\ &= e^{-2}\left[-\frac{8}{17} \cdot \frac{1}{p+2} + \frac{\frac{8}{17}p + \frac{1}{17}}{p^2 + \frac{1}{4}}\right] e^{-p} \\ &= \frac{e^{-2}}{17}\left[-8 \cdot \frac{1}{p+2} + 8 \cdot \frac{p}{p^2 + \frac{1}{4}} + 2 \cdot \frac{\frac{1}{2}}{p^2 + \frac{1}{4}}\right] e^{-p}.\end{aligned}$$

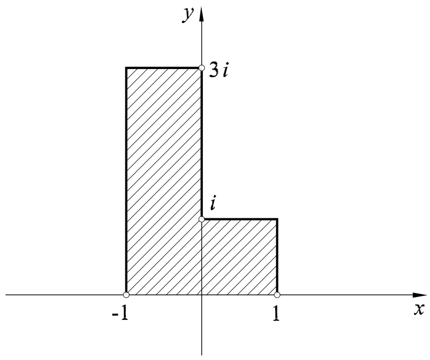
Ovako pripremljeno rješenje u donjem području lako vratimo u gornje područje.

$$\begin{aligned}\circ\!-\!\!\bigcirc\; i(t) = \frac{e^{-2}}{17}\Big[&-8e^{-2(t-1)}S(t-1) \\ &+ 8\cos\left(\frac{t-1}{2}\right)S(t-1) + 2\sin\left(\frac{t-1}{2}\right)S(t-1)\Big].\end{aligned}$$

Prva kontrolna zadaća

6. 12. 1997.

1. Područje sa slike preslikati funkcijom $f(z) = e^z$.



2. Funkciju

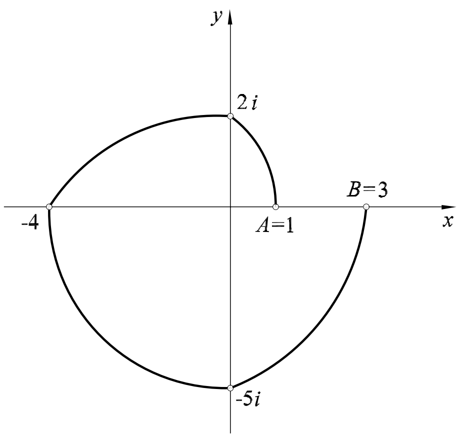
$$f(z) = \frac{1}{1 - z + z^2 - z^3 + z^4} + \sin \frac{\pi z - 3}{z}$$

razviti u red u okolini beskonačnosti. Izračunati koeficijent uz z^{-55} u tom razvoju.

3. Izračunati integral

$$\int_{\overset{\frown}{AB}} \frac{dz}{2 \operatorname{ch} z}$$

po luku $\overset{\frown}{AB}$ zadanom slikom.



Rješenja

6. 12. 1997.

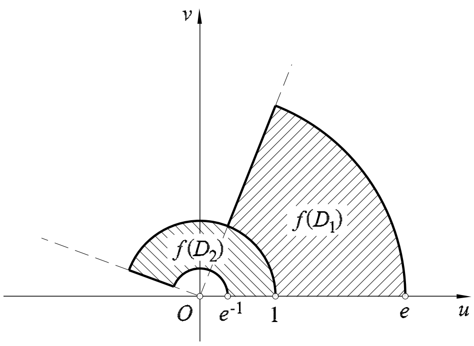
1.

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = R \cdot e^{i\Phi}$$
$$\implies R = e^x, \ \Phi = y$$

Područje sa slike možemo spretno podijeliti u dva pravokutnika koja se lagano parametriziraju i preslikaju zadanom funkcijom.

$$D_1 = \{z \in \mathbf{C} \mid x \in [0,1], \ y \in [0,1]\}$$
$$\implies R = e^x \in [1, e], \ \Phi = y \in [0,1]$$
$$D_2 = \{z \in \mathbf{C} \mid x \in [-1,0], \ y \in [0,3]\}$$
$$\implies R = e^x \in [e^{-1}, 1], \ \Phi = y \in [0,3]$$

Oba područja, i D_1 i D_2 su se preslikala u dijelove kružnih vijenaca. Uočimo da smo zadani šesterokut mogli preslikavati i krivulju po krivulju, no tada bi postupak bio dulji.



2. Zadanu funkciju moramo razviti po potencijama od $\dfrac{1}{z}$. Zato pišemo:

$$\begin{aligned}
f(z) &= \frac{1}{1-z+z^2-z^3+z^4} + \sin\left(\frac{\pi z - 3}{z}\right) \\
&= \frac{1}{1-z+z^2-z^3+z^4} \cdot \frac{1+z}{1+z} + \sin\left(\pi - \frac{3}{z}\right) \\
&= \frac{1+z}{1+z^5} + \sin\left(\frac{3}{z}\right) \\
&= (1+z) \cdot \frac{1}{z^5} \cdot \frac{1}{1-(-\frac{1}{z^5})} + \sin\left(\frac{3}{z}\right) \\
&= \frac{1+z}{z^5} \sum_{n=0}^{\infty} \left(-\frac{1}{z^5}\right)^n + \sin\left(\frac{3}{z}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{5n+5}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{5n+4}} + \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{3}{z})^{2n+1}}{(2n+1)!} \\
&= \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{5n+5}}}_{\substack{5n+5=55 \\ n=10, \ \text{postoji}}} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{5n+4}}}_{\substack{5n+4=55 \\ n=\frac{51}{5} \notin \mathbf{N}, \ \text{ne postoji}}} + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)! z^{2n+1}}}_{\substack{2n+1=55 \\ n=27, \ \text{postoji}}} .
\end{aligned}$$

Koeficijent uz z^{-55} moguće je dobiti u prvoj i trećoj sumi.

$$c_{-55} = (-1)^{10} + 0 + (-1)^{27} \cdot \frac{3^{55}}{55!} = 1 - \frac{3^{55}}{55!}.$$

3. Budući da luk $\overset{\frown}{AB}$ nije zadan, ne možemo integrirati po njemu po definiciji integrala funkcije kompleksne varijable, nego moramo zatvoriti zadanu krivulju i primijeniti Teorem o ostatcima. Krivulju zatvaramo na prirodan način, dužinom $\overline{BA}$. Uočimo da je ona pozitivno orijentirana.

Sada najprije transformiramo podintegralnu funkciju kako bismo joj odredili singularitete i u onima koje obuhvaća zatvorena krivulja izračunali reziduume (ostatke). Označimo s $I = \displaystyle\int_{\overset{\frown}{AB}} \frac{dz}{2 \operatorname{ch} z}$.

$$\begin{aligned}
f(z) &= \frac{1}{2 \operatorname{ch} z} = \frac{1}{e^z + e^{-z}} = \frac{e^z}{e^{2z}+1} \\
e^{2z} + 1 &= 0 \implies e^{2z} = -1 \implies 2z = \operatorname{Ln}(-1) \implies \\
z_k &= \frac{1}{2} \operatorname{Ln}(-1) = \frac{1}{2} [\underbrace{\ln|-1|}_{=0} + i \underbrace{\arg(-1)}_{=\pi} + 2k\pi] \\
&= \frac{i}{2}(\pi + 2k\pi), \quad k \in \mathbf{Z} \\
z_0 &= \frac{\pi}{2} i, \quad z_{-1} = -\frac{\pi}{2} i, \quad z_{-2} = -\frac{3\pi}{2} i.
\end{aligned}$$

Ova tri singulariteta, obuhvaćena zatvorenom krivuljom koju gledamo, očito su polovi 1. reda.

$$\begin{aligned}
\operatorname{Res}(f, z_k) &= \left[\frac{1}{(2 \operatorname{ch} z)'}\right]_{z=z_k} = \frac{1}{2 \operatorname{sh} z_k}. \\
\operatorname{Res}\left(f, z_0 = \frac{\pi}{2} i\right) &= \frac{1}{2 \operatorname{sh}(\frac{\pi}{2} i)} = \frac{1}{2i \sin \frac{\pi}{2}} = \frac{1}{2i}, \\
\operatorname{Res}\left(f, z_{-1} = -\frac{\pi}{2} i\right) &= \frac{1}{2 \operatorname{sh}(-\frac{\pi}{2} i)} = \frac{1}{-2i \sin \frac{\pi}{2}} = -\frac{1}{2i}, \\
\operatorname{Res}\left(f, z_{-2} = -\frac{3\pi}{2} i\right) &= \frac{1}{2 \operatorname{sh}(-\frac{3\pi}{2} i)} = \frac{1}{-2i \sin \frac{3\pi}{2}} = \frac{1}{2i}.
\end{aligned}$$

Po Teoremu o ostatcima vrijedi

$$I + \int_{\overline{BA}} \frac{dz}{2 \operatorname{ch} z} = 2\pi i \Bigg[\operatorname{Res}\left(f, \frac{\pi}{2} i\right) + \operatorname{Res}\left(f, -\frac{\pi}{2} i\right) + \operatorname{Res}\left(f, -\frac{3\pi}{2} i\right) \Bigg].$$

Zato je traženi integral

$$\begin{aligned}
I &= -\int_{\overline{BA}} \frac{dz}{2 \operatorname{ch} z} + 2\pi i \left(\frac{1}{2i} - \frac{1}{2i} + \frac{1}{2i}\right) \\
&= \int_{\overline{AB}} \frac{dz}{2 \operatorname{ch} z} + \pi.
\end{aligned}$$

Potrebno je još dakle po definiciji izračunati

$$\begin{aligned}
\int_{\overline{AB}} \frac{dz}{2 \operatorname{ch} z} &= \left| \begin{matrix} z = x \\ dz = dx \\ x \in [1,3] \end{matrix} \right| = \int_1^3 \frac{dx}{2 \operatorname{ch} x} \\
&= \int_1^3 \frac{dx}{e^x + e^{-x}} = \int_1^3 \frac{e^x \, dx}{(e^x)^2 + 1} = \int_1^3 \frac{d(e^x)}{(e^x)^2 + 1} \\
&= \operatorname{arc\,tg} e^x \Big|_{x=1}^{3} = \operatorname{arc\,tg} e^3 - \operatorname{arc\,tg} e.
\end{aligned}$$

Sveukupno rješenje zadatka je:

$$I = \int_{\overset{\frown}{AB}} \frac{dz}{2 \operatorname{ch} z} = \int_{\overline{AB}} \frac{dz}{2 \operatorname{ch} z} + \pi = \operatorname{arc\,tg} e^3 - \operatorname{arc\,tg} e + \pi.$$

Druga kontrolna zadaća

24. 1. 1998.

1. Izračunati konstantu $b \in \mathbf{R}$ tako da funkcije $f(x) = 1$ i $g(x) = |x| + b$ budu ortogonalne s obzirom na skalarni produkt definiran na intervalu $[-1, 2]$ s težinskom funkcijom $\rho(x) = e^{|1-x|}$.

2. Funkciju

$$f(x) = \begin{cases} \cos\left(\frac{\pi}{2}x\right), & |x| \leq 3 \\ 0, & \text{inače} \end{cases}$$

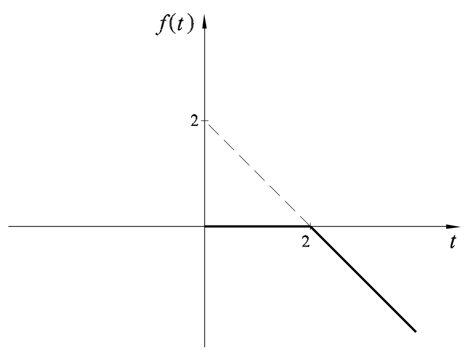
prikazati kao Fourierov integral. Odrediti amplitudni spektar te funkcije. Pomoću dobivenog prikaza izračunati integral

$$\int_0^\infty \frac{\cos 3u}{4u^2 - \pi^2}\, du.$$

3. Pomoću Laplaceove transformacije riješiti Cauchyjev problem

$$y''(t) - 6y'(t) + 10y(t) = f(t), \qquad y(0) = 0, \quad y'(0) = 1,$$

gdje je $f(t)$ funkcija zadana slikom.



Rješenja

24. 1. 1998.

1. Ispišimo najprije eksplicitnije zadane funkcije.
$x \in [-1, 2]$, $f(x) = 1$,

$$g(x) = |x| + b = \begin{cases} x+b, & x \geq 0 \\ -x+b, & x < 0 \end{cases}$$

$$\rho(x) = e^{|1-x|} = \begin{cases} e^{1-x}, & x \leq 1 \\ e^{x-1}, & x > 1 \end{cases}$$

Uvjet ortogonalnosti glasi:

$$\int_{-1}^{2} \underbrace{f(x)}_{=1} \cdot g(x) \cdot \rho(x)\, dx = 0.$$

Uvrštavanjem konkretnih funkcija i primjenom aditivnosti integrala dobivamo

$$\int_{-1}^{0} (-x+b)e^{1-x} dx + \int_{0}^{1} (x+b)e^{1-x} dx + \int_{1}^{2} (x+b)e^{x-1} dx = 0.$$

$$\begin{aligned}
\int_{-1}^{0} (-x+b)e^{1-x}\, dx &= \int_{-1}^{0} \underbrace{(-x+b)}_{u}\, d\underbrace{(-e^{1-x})}_{v} \\
&= (x-b)e^{1-x}\Big|_{x=-1}^{0} - \int_{-1}^{0} e^{1-x}\, dx \\
&= -be - (-1-b)e^2 + e^{1-x}\Big|_{x=-1}^{0} \\
&= -be + e^2 + be^2 + e - e^2 \\
&= -be + be^2 + e
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{1} (x+b)e^{1-x}\, dx &= \int_{0}^{1} \underbrace{(x+b)}_{u}\, d\underbrace{(-e^{1-x})}_{v} \\
&= -(x+b)e^{1-x}\Big|_{x=0}^{1} + \int_{0}^{1} e^{1-x}\, dx \\
&= -(1+b) + be - e^{1-x}\Big|_{0}^{1} \\
&= -1 - b + be - 1 + e \\
&= -2 - b + be + e
\end{aligned}$$

$$\begin{aligned}
\int_{1}^{2} (x+b)e^{x-1}\, dx &= \int_{1}^{2} \underbrace{(x+b)}_{u}\, d\underbrace{(e^{x-1})}_{v} \\
&= (x+b)e^{x-1}\Big|_{x=1}^{2} - \int_{1}^{2} e^{x-1}\, dx \\
&= (2+b)e - (1+b) - e^{x-1}\Big|_{1}^{2} \\
&= 2e + be - 1 - b - e + 1 \\
&= e + be - b.
\end{aligned}$$

Ukupno smo izračunali

$$-be + be^2 + e - 2 - b + be + e + e + be - b = 0$$

$$\Longrightarrow b = \frac{3e-2}{e^2+e-2}.$$

2. Zadana funkcija $f(x) = \begin{cases} \cos\left(\frac{\pi}{2}x\right), & |x| \leq 3 \\ 0, & \text{inače} \end{cases}$ je parna, pa je $B(\lambda) = 0$, a njen prikaz u Fourierov integral $f(x) = \int_0^\infty A(\lambda) \cdot \cos(\lambda x)\, d\lambda$.

Računamo kosinusni spektar

$$\begin{aligned}
A(\lambda) &= \frac{2}{\pi} \int_0^\infty f(\xi) \cdot \cos(\lambda\xi)\, d\xi \\
&= \frac{2}{\pi} \int_0^3 \cos\left(\frac{\pi}{2}\xi\right) \cdot \cos(\lambda\xi)\, d\xi \\
&= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^3 \left[\cos\left(\lambda + \frac{\pi}{2}\right)\xi + \cos\left(\lambda - \frac{\pi}{2}\right)\xi\right] d\xi \\
&= \frac{1}{\pi} \left[\frac{\sin(\lambda + \frac{\pi}{2})\xi}{\lambda + \frac{\pi}{2}} + \frac{\sin(\lambda - \frac{\pi}{2})\xi}{\lambda - \frac{\pi}{2}}\right]\Bigg|_{\xi=0}^{3} \\
&= \frac{1}{\pi} \left[\frac{\sin(3\lambda + \frac{3\pi}{2})}{\lambda + \frac{\pi}{2}} + \frac{\sin(3\lambda - \frac{3\pi}{2})}{\lambda - \frac{\pi}{2}}\right] \\
&= \frac{1}{\pi} \left[-\frac{\cos(3\lambda)}{\lambda + \frac{\pi}{2}} + \frac{\cos(3\lambda)}{\lambda - \frac{\pi}{2}}\right] \\
&= \frac{\cos(3\lambda)}{\lambda^2 - \frac{\pi^2}{4}} = \frac{4\cos(3\lambda)}{4\lambda^2 - \pi^2}.
\end{aligned}$$

Zbog neprekidnosti zadane funkcije smijemo je izjednačiti s njenim Fourierovim integralom

$$f(x) = \int_0^\infty \frac{4\cos(3\lambda)}{4\lambda^2 - \pi^2} \cdot \cos(\lambda x)\, d\lambda.$$

Traženi amplitudni spektar je

$$\text{am}\,(\lambda) = |A(\lambda)| = \left|\frac{4\cos(3\lambda)}{4\lambda^2 - \pi^2}\right| \text{ za } \lambda > 0 \text{ i } \lambda \neq \frac{\pi}{2}$$

$$\begin{aligned}
\text{am}\left(\frac{\pi}{2}\right) &= \left|\lim_{\lambda\to\frac{\pi}{2}} \frac{4\cos(3\lambda)}{4\lambda^2 - \pi^2}\right| = \left|\lim_{\lambda\to\frac{\pi}{2}} \frac{-12\sin(3\lambda)}{8\lambda}\right| \\
&= \left|\frac{12}{8 \cdot \frac{\pi}{2}}\right| = \frac{3}{\pi}
\end{aligned}$$

Dodatno zadani integral u zadatku izračunati ćemo pomoću dobivenog Fourierovog integrala za funkciju f, uvrstimo li točku $x = 0$ i iskoristimo $f(0) = 1$. Zato je

$$1 = f(0) = 4\int_0^\infty \frac{\cos(3\lambda)}{4\lambda^2 - \pi^2}\, d\lambda$$

$$\Longrightarrow \int_0^\infty \frac{\cos(3\lambda)}{4\lambda^2 - \pi^2}\, d\lambda = \frac{1}{4}$$

$$\Longrightarrow \int_0^\infty \frac{\cos(3u)}{4u^2 - \pi^2}\, du = \frac{1}{4}$$

3. Teorem o deriviranju originala nam daje:

$$y(t) \circ\!\!-\!\!\circ Y(p)$$

$$y'(t) \circ\!\!-\!\!\circ p \cdot Y(p) - y(0) = p \cdot Y(p)$$

$$y''(t) \circ\!\!-\!\!\circ p^2 \cdot Y(p) - p \cdot y(0) - y'(0) = p^2 \cdot Y(p) - 1$$

Funkciju f zadanu slikom zapišemo pomoću Step-funkcije te preslikamo Laplaceovom transformacijom ovako:

$$f(t) = (2-t)S(t-2) = -(t-2)S(t-2) \circ\!\!-\!\!\circ -\frac{1}{p^2}e^{-2p}.$$

Zato zadani Cauchyev problem u donjem području glasi

$$p^2Y(p) - 1 - 6pY(p) + 10Y(p) = -\frac{1}{p^2}e^{-2p}$$

$$(p^2 - 6p + 10)Y(p) = 1 - \frac{1}{p^2}e^{-2p}$$

$$Y(p) = \frac{1}{p^2 - 6p + 10} - \frac{1}{p^2(p^2 - 6p + 10)}e^{-2p}$$

$$\frac{1}{p^2(p^2-6p+10)} = \frac{A}{p} + \frac{B}{p^2} + \frac{Cp+D}{p^2-6p+10}$$
$$= \frac{3}{50} \cdot \frac{1}{p} + \frac{1}{10} \cdot \frac{1}{p^2} + \frac{-\frac{3}{50}p + \frac{13}{50}}{p^2-6p+10}$$

$$Y(p) = \frac{1}{(p-3)^2+1} - \left[\frac{3}{50} \cdot \frac{1}{p} + \frac{1}{10} \cdot \frac{1}{p^2} - \frac{3}{50} \cdot \frac{p-3}{(p-3)^2+1} + \frac{2}{25} \cdot \frac{1}{(p-3)^2+1}\right]e^{-2p}.$$

Ovako pripremljeno rješenje u donjem području lagano je vratiti u gornje područje, te dobivamo rješenje zadanog problema.

$$y(t) = e^{3t}\sin t \cdot S(t) - \frac{3}{50}S(t-2) - \frac{1}{10}(t-2) \cdot S(t-2)$$
$$+ \frac{3}{50}e^{3(t-2)}\cos(t-2) \cdot S(t-2)$$
$$- \frac{2}{25}e^{3(t-2)}\sin(t-2) \cdot S(t-2)$$

Prva kontrolna zadaća

28. 11. 1998.

1. Područje $D = \{z \in \mathbf{C} \mid |z| < 1,\ \operatorname{Im}(z) < 0,\ \operatorname{Re}(z) < 0\}$ preslikati funkcijom

$$w = f(z) = \left[\frac{(-1-i)z - 1 + i}{z+i}\right]^3.$$

Skicirati D, $w(D)$, te napisati jednadžbe rubnih krivulja područja $w(D)$.

2. Pomoću Cauchyjeve integralne formule izračunati

$$\oint_{|z|=3} \frac{dz}{z^2 + 2i}.$$

3. Izračunati

$$\oint_{|z|=1} (z-1)^2 \left[\cos\frac{1}{2z-i} + \frac{\cos(2z-i)}{z-1}\right] dz.$$

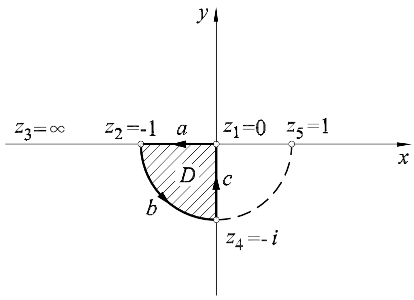
Rješenja

28. 11. 1998.

1. Zadanu funkciju w shvatimo kao kompoziciju dviju jednostavnijih funkcija

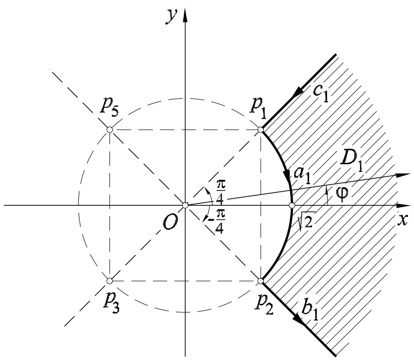
$$w = f(z) = \left[\frac{(-1-i)z-1+i}{z+i}\right]^3 \implies$$
$$p(z) = \frac{(-1-i)z-1+i}{z+i}, \quad w = p^3.$$

Prvo preslikavamo Möbiusovom transformacijom p, i zato biramo sljedeće točke, te u njima računamo vrijednost funkcije $p(z)$.



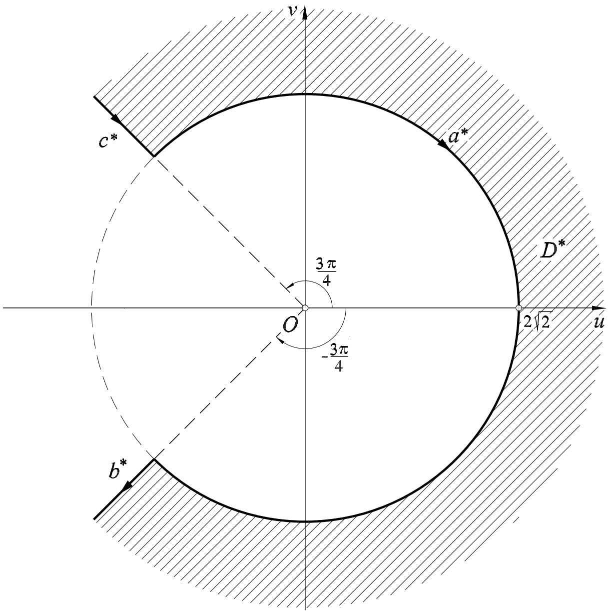
$$\begin{aligned}
z_1 &= 0 & &\rightarrow & p_1 &= 1+i\\
z_2 &= -1 & &\rightarrow & p_2 &= 1-i\\
z_3 &= \infty & &\rightarrow & p_3 &= -1-i\\
z_4 &= -i & &\rightarrow & p_4 &= \infty\\
z_5 &= 1 & &\rightarrow & p_5 &= -1+i
\end{aligned}$$

Rješenje je dano slikom. Preostaje preslikati ovo područje $p(D) = D_1$ funkcijom $w = p^3$.



$$p = r\cdot e^{i\varphi},\ r \in [\sqrt{2}, +\infty\rangle,\ \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$
$$w = p^3 = (re^{i\varphi})^3 = r^3\cdot e^{i3\varphi} = R\cdot e^{i\Phi}$$
$$R = |w| = r^3 \in [2\sqrt{2}, +\infty\rangle,$$
$$\Phi = \arg w = 3\varphi \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right].$$

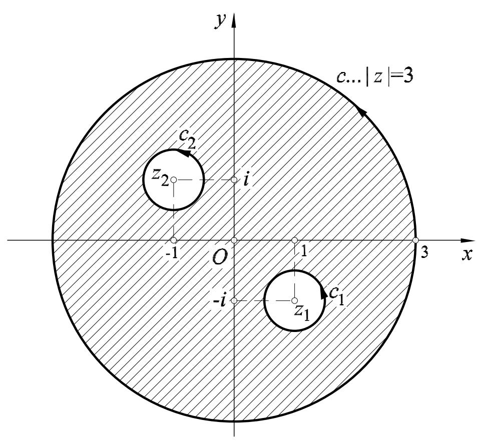
Dobiveni skup $w(D)$ ovime je u potpunosti determiniran i dan je sljedećom slikom koja je konačno rješenje zadatka.



2. Označimo s $I = \displaystyle\oint\limits_{|z|=3} \frac{dz}{z^2+2i}$. Tražimo nultočke nazivnika podintegralne funkcije.

$$\begin{aligned}
&z^2 + 2i = 0 \implies\\
&z_{1,2} = \sqrt{-2i} = \sqrt{2}\cdot\ \mathrm{cis}\left(\frac{-\frac{\pi}{2}+2k\pi}{2}\right)\\
&\quad = \sqrt{2}\cdot\ \mathrm{cis}\left(-\frac{\pi}{4}+k\pi\right),\ k = 0, 1\\
&z_1 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]\\
&\quad = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 1-i,\\
&z_2 = -z_1 = -1+i.
\end{aligned}$$

Za obje nultočke vrijedi $|z_i| < 3$, zato razdvajamo integral i primjenjujemo Cauchyevu integralnu formulu na svaki od njih.



$$
\begin{aligned}
I &= \oint_{C\dots\ |z|=3} \frac{dz}{z^2+2i} \\
&= \oint_C \frac{1}{[z-(1-i)][z-(-1+i)]}\,dz \\
&= \oint_{C_1} \frac{\frac{1}{z-(-1+i)}}{z-(1-i)}\,dz + \oint_{C_2} \frac{\frac{1}{z-(1-i)}}{z-(-1+i)}\,dz \\
&= 2\pi i\left[\frac{1}{z-(-1+i)}\right]\Big|_{z_1=1-i} + 2\pi i\left[\frac{1}{z-(1-i)}\right]\Big|_{z_2=-1+i} \\
&= 2\pi i\left[\frac{1}{1-i+1-i} + \frac{1}{-1+i-1+i}\right] \\
&= 2\pi i\left[\frac{1}{2-2i} - \frac{1}{2-2i}\right] = 0.
\end{aligned}
$$

3. Najprije izmnožimo podintegralnu funkciju i izanaliziramo njene tako dobivene pribrojnike.

$$
\begin{aligned}
I &= \oint_{|z|=1} (z-1)^2\left[\cos\left(\frac{1}{2z-i}\right) + \frac{\cos(2z-i)}{z-1}\right]dz \\
&= \oint_{|z|=1} (z-1)^2\cos\left(\frac{1}{2z-i}\right)dz \\
&\quad + \underbrace{\oint_{|z|=1} \underbrace{(z-1)\cos(2z-i)\,dz}_{\text{analitička } \forall z\in\mathbf{C}}}_{=0} \\
&= \oint_{|z|=1} \underbrace{(z-1)^2\cos\left(\frac{1}{2z-i}\right)}_{=f(z)}dz.
\end{aligned}
$$

Ostaje nam promotriti singularitete ove podintegralne funkcije.

$$
f(z) = (z-1)^2\cos\left(\frac{1}{2z-i}\right),\ 2z-i=0 \implies z=\frac{i}{2}
$$

je bitno singularna točka, pa moramo funkciju $f(z)$ razviti u njenoj okolini u Laurentov red.

$$
\begin{aligned}
f(z) &= (z-1)^2\cos\left(\frac{1}{2z-i}\right) \\
&= \left[\left(z-\frac{i}{2}\right) + \left(\frac{i}{2}-1\right)\right]^2 \cos\left[\frac{1}{2(z-\frac{i}{2})}\right] \\
&= \left[\left(z-\frac{i}{2}\right)^2 + 2\left(\frac{i}{2}-1\right)\left(z-\frac{i}{2}\right) + \left(\frac{i}{2}-1\right)^2\right] \\
&\quad \cdot \sum_{n=0}^{\infty}(-1)^n \frac{\left[\frac{1}{2(z-\frac{i}{2})}\right]^{2n}}{(2n)!} \\
&= \left[\left(z-\frac{i}{2}\right)^2 + 2\left(\frac{i}{2}-1\right)\left(z-\frac{i}{2}\right) + \left(\frac{i}{2}-1\right)^2\right] \\
&\quad \cdot \left\{1 - \frac{1}{2!}\cdot\frac{1}{2^2(z-\frac{i}{2})^2} + \frac{1}{4!}\cdot\frac{1}{2^4(z-\frac{i}{2})^4} + \dots\right\} \\
&= \dots + \frac{1}{z-\frac{i}{2}}\left[-\frac{2(\frac{i}{2}-1)}{2!\cdot 2^2}\right] + \dots \\
&= \dots + \frac{1}{z-\frac{i}{2}} \cdot \underbrace{\left(\frac{2-i}{8}\right)}_{=c_{-1}=\,\mathrm{Res}\,(f(z),\frac{i}{2})} + \dots .
\end{aligned}
$$

Uspjeli smo pročitati reziduum iz ovog Laurentovog razvoja, pa primjena Teorema o reziduumima odmah daje konačan rezultat:

$$
\begin{aligned}
I &= 2\pi i\cdot\ \mathrm{Res}\left(f(z),\frac{i}{2}\right) = 2\pi i\left(\frac{2-i}{8}\right) \\
&= \frac{2-i}{4}\pi i = \frac{1+2i}{4}\pi.
\end{aligned}
$$

Druga kontrolna zadaća

20. 1. 1999.

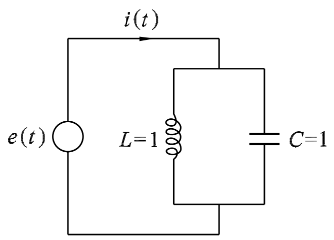
1. Funkciju $f(x) = \arccos x$ razviti u red po Čebiševljevim polinomima na intervalu $[-1, 1]$, te pomoću tog razvoja izračunati sumu reda:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

2. Pomoću Laplaceove transformacije izračunati integral:

$$\int_0^{\infty} e^{-7x} \cdot x \cdot \sin x \, dx.$$

3. Pomoću Laplaceove transformacije naći struju $i(t)$ električnog kruga zadanog slikom uz priključeni napon $e(t) = 2\cos^2 t$.



Rješenja

20. 1. 1999.

1. Zadanu funkciju

$$f(x) = \arccos x,\ x \in [-1, 1],$$

razviti u red po Čebiševljevim polinomima

$$f(x) = c_0 T_0(x) + \ldots + c_n T_n(x) + \ldots$$

znači izračunati pripadne koeficijente c_n.

$$\begin{aligned}
c_0 &= \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \\
&= \frac{1}{\pi}\int_{-1}^{1} \frac{\arccos x}{\sqrt{1-x^2}}\,dx \\
&= -\frac{1}{\pi}\int_{-1}^{1} \arccos x \, d(\arccos x) \\
&= -\frac{1}{\pi}\cdot \frac{(\arccos x)^2}{2}\Big|_{x=-1}^{1} \\
&= -\frac{1}{2\pi}[(\arccos 1)^2 - (\arccos(-1))^2] \\
&= -\frac{1}{2\pi}[0-\pi^2] = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
c_n &= \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,dx \\
&= \frac{2}{\pi}\int_{-1}^{1} \frac{\arccos x T_n(x)}{\sqrt{1-x^2}}\,dx \\
&= \left| \begin{array}{c} x = \cos t \\ dx = -\sin t\, dt \\ x=-1 \implies t=\pi \\ x = 1 \implies t = 0 \end{array} \right| \\
&= \frac{2}{\pi}\int_{\pi}^{0} \frac{\arccos(\cos t)T_n(\cos t)}{\sqrt{1-\cos^2 t}}(-\sin t)\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\cdot \cos(nt)\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} \underbrace{t}_{u}\, d\underbrace{\left(\frac{\sin(nt)}{n}\right)}_{v} \\
&= \frac{2}{\pi}\left[\underbrace{t\frac{\sin(nt)}{n}\Big|_{t=0}^{\pi}}_{=0,\ \forall n\in\mathbf{N}} - \int_0^{\pi} \frac{\sin(nt)}{n}\,dt\right] \\
&= \frac{2}{\pi}\cdot\frac{\cos(nt)}{n^2}\Big|_{t=0}^{\pi} = \frac{2}{\pi n^2}[\cos(n\pi)-1]
\end{aligned}$$

$$c_n = \frac{2}{\pi n^2}[(-1)^n - 1],\ n = 1, 2, 3, \ldots \implies$$

$$c_{2n} = 0, \quad c_{2n+1} = -\frac{4}{\pi(2n+1)^2},\ n = 0, 1, 2, \ldots$$

$$\arccos x = \frac{\pi}{2}T_0(x) - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}T_{2n+1}(x).$$

Da bismo izračunali traženu sumu brojeva, uvrstimo $x = 1$ u gornju jednakost, pa dobivamo:

$$\begin{aligned}
&\underbrace{\arccos 1}_{=0} = \frac{\pi}{2}\underbrace{T_0(1)}_{=1} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\underbrace{T_{2n+1}(1)}_{=1} \\
&\implies 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&\implies \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.
\end{aligned}$$

2. $\displaystyle\int_0^{\infty} e^{-7x}\cdot x\cdot \sin x\,dx = F(7)$, gdje smo s F označili

$$F(p) = \int_0^{\infty} e^{-px}\cdot x\cdot\sin x\,dx = \mathcal{L}\{x\cdot\sin x\}$$

$$\sin x \circ\!\!-\!\!\circ \frac{1}{p^2+1}.$$

Teorem o deriviranju slike daje

$$x\sin x \circ\!\!-\!\!\circ (-1)\cdot\left(\frac{1}{p^2+1}\right)' = \frac{2p}{(p^2+1)^2} = F(p)$$

$$F(p) = \frac{2p}{(p^2+1)^2}$$

$$F(7) = \frac{14}{50^2} = \frac{14}{2500} = \frac{7}{1250}$$

$$\implies \int_0^{\infty} e^{-7x}\cdot x\cdot\sin x\,dx = \frac{7}{1250}.$$

3. Najprije transformiramo i preslikamo zadanu naponsku funkciju.

$$e(t) = 2\cos^2 t\cdot S(t) = [1+\cos(2t)]\cdot S(t)$$

$$\circ\!\!-\!\!\circ\ E(p) = \frac{1}{p} + \frac{p}{p^2+2^2} = \frac{1}{p} + \frac{p}{p^2+4}.$$

Ukupni otpor u donjem području je

$$Z(p) = \frac{p\cdot\frac{1}{p}}{p+\frac{1}{p}} = \frac{p}{p^2+1}.$$

Slijedi

$$\begin{aligned}
I(p) &= \frac{E(p)}{Z(p)} = \frac{p^2+1}{p}\left[\frac{1}{p}+\frac{p}{p^2+4}\right] \\
&= \frac{p^2+1}{p^2} + \frac{p^2+1}{p^2+4} \\
&= 1 + \frac{1}{p^2} + \frac{(p^2+4)-3}{p^2+4} \\
&= 1 + \frac{1}{p^2} + 1 - \frac{3}{p^2+4} \\
&= 2 + \frac{1}{p^2} - \frac{3}{2}\cdot\frac{2}{p^2+2^2}\ \circ\!\!-\!\!\circ
\end{aligned}$$

$$i(t) = 2\cdot\delta(t) + t\cdot S(t) - \frac{3}{2}\cdot\sin(2t)\cdot S(t).$$

Prva kontrolna zadaća

27. 11. 1999.

1. Područje

$$D = \left\{ z \in \mathbf{C} \mid 1 < |z| < 2,\ \frac{3\pi}{4} < \arg(z) < \frac{3\pi}{2} \right\}$$

preslikati funkcijom

$$w(z) = z - \frac{4}{z}.$$

U kojoj točki ruba područja D preslikavanje nije konformno? Skicirati D, $w(D)$, te napisati jednadžbe rubnih krivulja područja $w(D)$ u implicitnom obliku.

2. Funkciju

$$f(z) = \frac{4}{z^2 + 2i}$$

razviti u Laurentov red oko točke $z_0 = -i$, tako da područje konvergencije sadrži točku $z_1 = i$. Naći i skicirati dotično područje.

3. Izračunati

$$I = \oint\limits_{|z+i|=3} \frac{\operatorname{tg} z}{e^{2z} - 1}\, dz.$$

Rezultat prikazati u algebarskom obliku $I = x + iy$. Kojeg su tipa singulariteti podintegralne funkcije unutar zadane zatvorene pozitivno orijentirane krivulje?

Rješenja

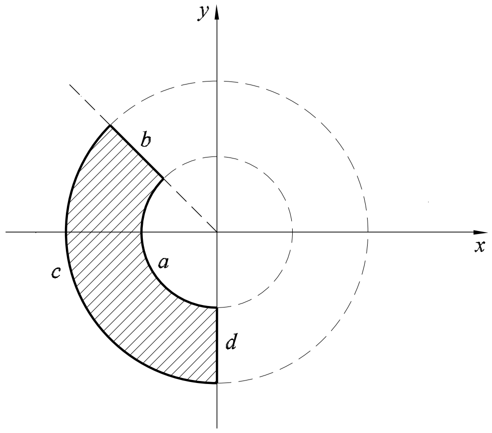
27. 11. 1999.

1.

$$w(z) = z - \frac{4}{z}$$

$$w'(z) = 1 + \frac{4}{z^2} = 0 \implies z_{1,2} = \pm 2i$$

Zaključujemo da su točke 0 (u kojoj zadana funkcija nije ni definirana), $2i$, te $-2i$ jedini kandidati za točke nekonformnosti. Uočimo da je zapravo jedini kandidat točka $-2i$, jer jedino ona leži na rubu zadanog područja D. Preslikajmo sada rubne krivulje područja D jednu po jednu.



a) $|z| = 1 \implies z = e^{i\varphi},\ \varphi \in \left[\frac{3\pi}{4}, \frac{3\pi}{2}\right]$,

$$\begin{aligned} w(z) &= e^{i\varphi} - \frac{4}{e^{i\varphi}} \\ &= \cos\varphi + i\sin\varphi - (4\cos\varphi - 4i\sin\varphi) \\ &= -3\cos\varphi + 5i\sin\varphi \\ &\implies u = -3\cos\varphi,\ v = 5\sin\varphi,\ \varphi \in \left[\frac{3\pi}{4}, \frac{3\pi}{2}\right] \\ &\implies \frac{u^2}{9} + \frac{v^2}{25} = 1, \end{aligned}$$

tj. dobiva se dio ove elipse.

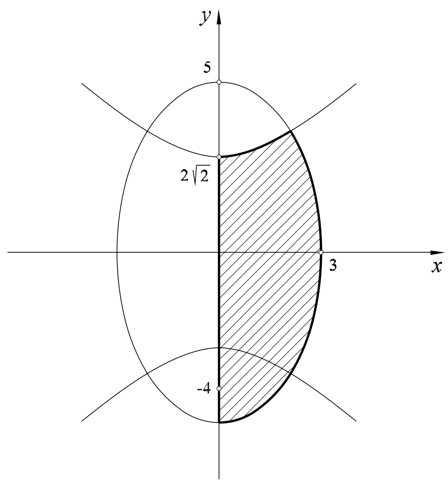
b) $y = -x,\ x \in \left[-\sqrt{2}, -\frac{\sqrt{2}}{2}\right]$,

$$\begin{aligned} w(x+iy) &= x + iy - \frac{4}{x+iy} \\ &\overset{b}{=} x - ix - \frac{4}{x - ix} \\ &= x - ix - \frac{2}{x}(1+i) \\ &= x - \frac{2}{x} - i\left(x + \frac{2}{x}\right) \\ &\implies u = x - \frac{2}{x},\ v = -x - \frac{2}{x} \\ &\implies u - v = 2x,\ x = \frac{u-v}{2} \\ &\implies u = \frac{u-v}{2} - \frac{4}{u-v} \\ &\implies u^2 - v^2 = -8 \\ &\implies -\frac{u^2}{8} + \frac{v^2}{8} = 1. \end{aligned}$$

Dakle slika ove dužine je dio okrenute hiperbole.

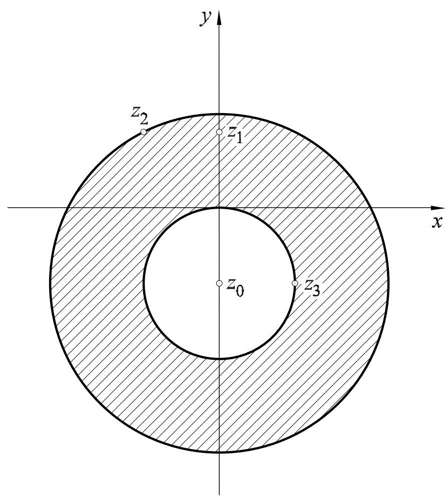
c) $z = 2e^{i\varphi},\ \varphi \in \left[\frac{3\pi}{4}, \frac{3\pi}{2}\right] \implies w(z) = 2e^{i\varphi} - 2e^{-i\varphi} = 4i\sin\varphi \implies u = 0,\ v = 4\sin\varphi,\ \varphi \in \left[\frac{3\pi}{4}, \frac{3\pi}{2}\right]$. Ovdje je slika zadanog kružnog luka dio pravca $u = 0$.

d) $x = 0,\ w = iy - \frac{4}{iy} = i\left(y + \frac{4}{y}\right) \implies u = 0, v = y + \frac{4}{y}$, $y \in [-2, -1]$. I ovdje je riječ o dijelu istog pravca kao kod preslikavanja prethodne krivulje.



Uočimo zaključno da u slici točke $-2i$, $w(-2i) = -2i - \frac{4}{-2i} = -4i$ kut nije sačuvan (u originalu D on iznosi $\pi/2$ a u slici $w(D)$ 0), pa preslikavanje u točki $z_0 = -2i$ nije konformno.

2. Singulariteti zadane funkcije su $z^2 = -2i \implies z_2 = -1 + i, z_3 = 1 - i$.



Uočimo da je $|z_3 - z_0| < |z_1 - z_0| < |z_2 - z_0|$, pa ćemo razvijati funkciju f u području $1 < |z + i| < \sqrt{5}$.

$$\begin{aligned} \frac{4}{z^2 + 2i} &= \frac{4}{(z+1-i)(z-1+i)} \\ &= \frac{A}{z+1-i} + \frac{B}{z-1+i} \\ &= \ldots = \frac{-1-i}{z+1-i} + \frac{1+i}{z-1+i} \end{aligned}$$

$$
\begin{aligned}
\frac{-1-i}{z+1-i} &= \frac{-1-i}{(z+i)+(1-2i)} \\
&= \frac{-1-i}{(1-2i)(1+\frac{z+i}{1-2i})} \\
&= \frac{-1-i}{1-2i}\sum_{n=0}^{\infty}(-1)^n\frac{(z+i)^n}{(1-2i)^n} \\
&= (1+i)\sum_{n=0}^{\infty}\left(\frac{-1}{1-2i}\right)^{n+1}(z+i)^n \\
\frac{1+i}{z-1+i} &= \frac{1+i}{(z+i)(1-\frac{1}{z+i})} \\
&= \frac{1+i}{z+i}\sum_{n=0}^{\infty}\frac{1}{(z+i)^n} \\
&= (1+i)\sum_{n=0}^{\infty}\frac{1}{(z+i)^{n+1}}.
\end{aligned}
$$

Konačno, dobiveni Laurentov razvoj je

$$
\begin{aligned}
f(z) = (1+i)\sum_{n=0}^{\infty}\left(\frac{-1}{1-2i}\right)^{n+1}(z+i)^n \\
+ (1+i)\sum_{n=0}^{\infty}\frac{1}{(z+i)^{n+1}}.
\end{aligned}
$$

3. Nultočke svih nazivnika zadane podintegralne funkcije, obuhvaćene zadanom kružnicom su $e^{2z}-1=0 \implies e^{2z}=1 \implies 2z=2k\pi i \implies z=k\pi i \implies z_0=0,\ z_1=-\pi i$; $\cos z=0 \implies z=\frac{\pi}{2}+k\pi,\ z_2=\frac{\pi}{2},\ z_3=-\frac{\pi}{2}$. Lako se vidi da je $z_0=0$ uklonjivi singularitet, a da su $z_{1,2,3}$ polovi 1. reda. U polovima 1. reda računamo reziduume:

$$
\begin{aligned}
\operatorname{Res}(f,-\pi i) &= \left.\frac{\operatorname{tg} z}{2e^{2z}}\right|_{z_1=-\pi i} = \frac{\operatorname{tg}(-\pi i)}{2} = -\frac{i}{2}\operatorname{th}\pi \\
\operatorname{Res}\left(f,\frac{\pi}{2}\right) &= \left.\frac{\sin z}{-\sin z(e^{2z}-1)+\cos z\cdot 2e^{2z}}\right|_{z_2=\frac{\pi}{2}} \\
&= \frac{1}{1-e^{\pi}} \\
\operatorname{Res}\left(f,-\frac{\pi}{2}\right) &= \left.\frac{\sin z}{-\sin z(e^{2z}-1)+\cos z\cdot 2e^{2z}}\right|_{z_3=-\frac{\pi}{2}} \\
&= \frac{1}{1-e^{-\pi}}.
\end{aligned}
$$

Po Teoremu o reziduumima je sada

$$
\begin{aligned}
I &= 2\pi i\left[-\frac{i}{2}\operatorname{th}\pi+\frac{1}{1-e^{\pi}}+\frac{1}{1-e^{-\pi}}\right] \\
&= \pi\operatorname{th}\pi + i\left(\frac{1}{1-e^{\pi}}+\frac{1}{1-e^{-\pi}}\right)2\pi \\
&= \pi\operatorname{th}\pi + 2\pi i.
\end{aligned}
$$

Pismeni ispit

4. 2. 1997.

1. Imaginarni dio Möbiusove transformacije $w = u + iv$, za koju je $w(1) = 3$, iznosi $v(x, y) = -\frac{2y}{x^2 + y^2}$. Odrediti sliku $w(D)$ područja $D = \{z \in \mathbf{C} \mid \mathrm{Re}\,(z) < 0,\ \mathrm{Im}\,(z) > 0,\ |z| < 1\}$ po preslikavanju w i skicirati područje D i njegovu sliku $w(D)$.

2. Izračunati

$$\oint\limits_{|z|=3} (z-1)^2 e^{\frac{1}{z-2}}\, dz.$$

3. Pomoću prikaza funkcije

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & -1 < x < 0 \\ 0, & \text{inače} \end{cases}$$

u obliku Fourierovog integrala izračunati

$$\int_0^\infty \frac{\sin^3 x}{x}\, dx.$$

4. Riješiti jednadžbu:

$$y'(t) - 4\int_0^t y(u)\, du + 6e^t \int_0^t \frac{y(u)}{e^u}\, du = 0, \quad y(0) = 1.$$

5. Naći opće rješenje diferencijalne jednadžbe

$$y'' + 4y = \sin x + \cos 2x.$$

Rješenja

4. 2. 1997.

1.

$$v = -\frac{2y}{x^2+y^2}.$$

Zbog Cauchy-Riemannovih uvjeta je

$$\begin{aligned} du &= v'_y\,dx - v'_x\,dy \\ &= \frac{2(y^2-x^2)}{(x^2+y^2)^2}\,dx - \frac{4xy}{(x^2+y^2)^2}\,dy, \text{ pa je} \\ u &= \int\limits_{x_0=0}^{x} \frac{2(y^2-x^2)}{(x^2+y^2)^2}\,dx - \int\limits_{y_0=1}^{y} \underbrace{\frac{4x_0y}{(x_0^2+y^2)^2}\,dy}_{=0} + C \\ &= 2\int\limits_0^x \frac{dx}{x^2+y^2} - 4\int\limits_0^x x\,d\left(\frac{-1}{2(x^2+y^2)}\right) \\ u &= \frac{2x}{x^2+y^2} + C \\ w(1) &= 3 \implies C = 1. \end{aligned}$$

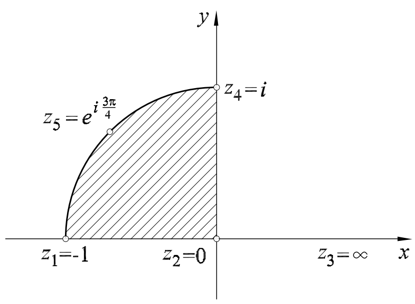
Zaključno, tražena Möbiusova transformacija je

$$w = \frac{2x}{x^2+y^2} + 1 - i\frac{2y}{x^2+y^2}$$

odnosno

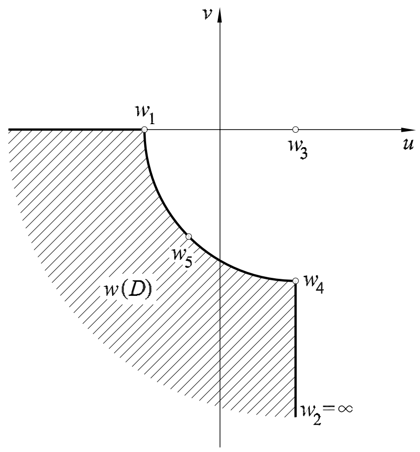
$$w = \frac{2}{z} + 1.$$

Sada biramo po tri točke sa svake rubne krivulje područja D, što je moguće ostvariti s ukupno sljedećih pet točaka, u kojima računamo vrijednost transformacije $w(z)$.



$$\begin{aligned} z_1 &= -1 && \to && w_1 = -1 \\ z_2 &= 0 && \to && w_2 = \infty \\ z_3 &= \infty && \to && w_3 = 1 \\ z_4 &= i && \to && w_4 = 1-2i \\ z_5 &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} && \to && w_5 = (1-\sqrt{2}) - i\sqrt{2} \end{aligned}$$

Dobivene slike leže na kružnicama, pa rješenje, dano slikom, neposredno slijedi.



2. Jedina singularna točka podintegralne funkcije je $z_0 = 2$, a to je bitni singularitet. Zato podintegralnu funkciju razvijamo u Laurentov red u njenoj okolini.

$$\begin{aligned} (z-1)^2 e^{\frac{1}{z-2}} &= [(z-2)+1]^2 e^{\frac{1}{z-2}} \\ &= [(z-2)^2 + 2(z-2) + 1] \\ &\cdot \left[1 + \frac{1}{z-2} + \frac{1}{2}\cdot\frac{1}{(z-2)^2} + \frac{1}{6}\cdot\frac{1}{(z-2)^3} + \ldots\right] \\ \operatorname{Res}(f,2) &= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{2} + 1\cdot 1 = \frac{13}{6}. \end{aligned}$$

Po Teoremu o reziduumima je sada

$$I = 2\pi i\frac{13}{6} = \frac{13}{3}\pi i.$$

3. Zadana funkcija f je neparna, pa je $A(\lambda) = 0$. Zato je

$$\begin{aligned} f(x) &= \int\limits_0^\infty B(\lambda)\sin\lambda x\,d\lambda \\ B(\lambda) &= \frac{2}{\pi}\int\limits_0^1 1\cdot\sin\lambda x\,dx \\ &= -\frac{2\cos\lambda x}{\lambda\pi}\Big|_{x=0}^{1} = \frac{2}{\pi}\cdot\frac{1-\cos\lambda}{\lambda} \\ f(x) &= \frac{2}{\pi}\int\limits_0^\infty \frac{1-\cos\lambda}{\lambda}\cdot\sin\lambda x\,d\lambda \\ &= \frac{4}{\pi}\int\limits_0^\infty \frac{\sin^2\frac{\lambda}{2}}{\lambda}\cdot\sin\lambda x\,d\lambda \end{aligned}$$

Uvrštavanjem konkretne točke $x = \frac{1}{2}$ slijedi:

$$\begin{aligned} 1 &= \frac{4}{\pi}\int\limits_0^\infty \frac{\sin^2\frac{\lambda}{2}}{\lambda}\cdot\sin\frac{\lambda}{2} \\ \frac{\pi}{4} &= \int\limits_0^\infty \frac{\sin^3\frac{\lambda}{2}}{\frac{\lambda}{2}}\,d\left(\frac{\lambda}{2}\right) \\ I &= \frac{\pi}{4} \end{aligned}$$

4. Prepoznajemo da zadanu diferencijalno-integralnu jednadžbu možemo zapisati u obliku

$$y'(t) - 4(1 * y(t)) + 6(e^t * y(t)) = 0$$

što preslikano u donje područje daje

$$pY(p) - 1 - 4\left(\frac{1}{p} \cdot Y(p)\right) + 6\left(\frac{1}{p-1} \cdot Y(p)\right) = 0$$

$$\begin{aligned}
Y(p)\left[p - \frac{4}{p} + \frac{6}{p-1}\right] &= 1 \\
Y(p) &= \frac{p^2 - p}{p^3 - p^2 + 2p + 4} \\
&= \frac{A}{p+1} + \frac{Bp + C}{p^2 - 2p + 4} \\
&= \ldots\ A = \frac{2}{7},\ B = \frac{5}{7},\ C = -\frac{8}{7},\ \ldots \\
&= \frac{\frac{2}{7}}{p+1} + \frac{\frac{5}{7}p - \frac{8}{7}}{p^2 - 2p + 4} \\
&= \frac{2}{7} \cdot \frac{1}{p+1} + \frac{5}{7} \cdot \frac{p-1}{(p-1)^2 + 3} + \frac{\sqrt{3}}{7} \cdot \frac{\sqrt{3}}{(p-1)^2 + 3}
\end{aligned}$$

$$\circ\!\!-\!\!\bigcirc\ y(t) = \frac{2}{7}e^{-t} + \frac{5}{7}\cos(\sqrt{3}t)e^t - \frac{\sqrt{3}}{7}\sin(\sqrt{3}t)e^t$$

5. Najprije rješavamo pripadnu homogenu jednadžbu

$$\begin{aligned}
y'' + 4y &= 0 \\
\lambda^2 + 4 &= 0 \\
\lambda_{1,2} &= \pm 2i \\
y_0 &= c_1 \cos 2x + c_2 \sin 2x.
\end{aligned}$$

Dobili smo opće rješenje homogene jednadžbe. Za svaki od pribrojnika s desne strane zadane jednadžbe odvojeno moramo metodom neodređenih koeficijenata tražiti partikularno rješenje.

$$\begin{aligned}
&y_{p_1} = A\cos x + B\sin x \\
&\Longrightarrow -A\cos x - B\sin x + 4A\cos x + 4B\sin x = \sin x \\
&\Longrightarrow A = 0,\ 3B = 1,\ B = \frac{1}{3} \\
&y_{p_1} = \frac{1}{3}\sin x \\
&y_{p_2} = Cx\cos 2x + Dx\sin 2x \\
&\Longrightarrow -4C\sin 2x + 4D\cos 2x - 4Cx\cos 2x - 4Dx\sin 2x \\
&\qquad + 4(Cx\cos 2x + Dx\sin 2x) = \cos 2x \\
&\Longrightarrow C = 0,\ D = \frac{1}{4} \\
&y_{p_2} = \frac{1}{4}x\sin 2x.
\end{aligned}$$

Rješenje zadane diferencijalne jednadžbe zbroj je općeg rješenja pripadne homogene jednadžbe i oba nađena partikularna rješenja.

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\sin x + \frac{1}{4}x\sin 2x.$$