

Prva kontrolna zadaća

24. 11. 2000.

1. Naći ona rješenja jednadžbe $(1 - i)z^4 + (1 + i)z = 0, z \in \mathbf{C}$ koja zadovoljavaju uvjet $|z - 1 + i| \leq \frac{3}{2}$.

2. Odrediti domenu (područje definicije) funkcije:

$$f(x) = \sqrt{x + 1 + \operatorname{sh}(\ln x)}.$$

3. Izračunati:

$$\lim_{n \rightarrow \infty} \frac{1 + 5^n}{1 + 5 + 5^2 + \dots + 5^{n-1} + 5^n}.$$

4. Ispitati konvergenciju reda:

a) $\sum_{n=1}^{\infty} \frac{1}{2n + \sqrt{n}};$

b) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n \cdot n!}.$

Rješenja

24. 11. 2000.

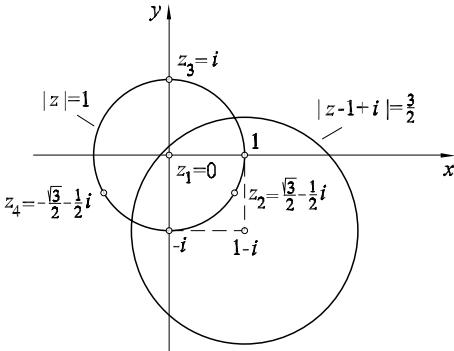
1. $(1-i)z^4 + (1-i)z = 0, z \in \mathbf{C}, |z - 1+i| \leq \frac{3}{2}$
 $z[(1-i)z^3 + (1+i)] = 0 \implies$

a) $z_1 = 0$

b) $z = \sqrt[3]{-\frac{1+i}{1-i}} = \sqrt[3]{-i}$
 $= \sqrt[3]{|1-i|} \left[\cos\left(\frac{\arg(-i) + 2k\pi}{3}\right) + i \sin\left(\frac{\arg(-i) + 2k\pi}{3}\right) \right]$
 $= \sqrt[3]{1} \left[\cos\left(\frac{-\pi/2 + 2k\pi}{3}\right) + i \sin\left(\frac{-\pi/2 + 2k\pi}{3}\right) \right]$
 $= \cos\left(-\frac{\pi}{6} + k \cdot \frac{2\pi}{3}\right) + i \sin\left(-\frac{\pi}{6} + k \cdot \frac{2\pi}{3}\right),$
 $k = 0, 1, 2$
 $z_2 = \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)$
 $= \cos\frac{\pi}{6} - i \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$
 $z_3 = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i$
 $z_4 = \cos\left(\frac{7\pi}{6}\right) + i \sin\left(\frac{7\pi}{6}\right)$
 $= -\cos\frac{\pi}{6} - i \sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$

Uvjet $|z - 1+i| \leq \frac{3}{2}$ zadovoljavaju:

$$\begin{aligned} z_1 &= 0 \implies |0 - 1+i| \leq \frac{3}{2} \implies \sqrt{2} \leq \frac{3}{2} \\ z_2 &= \frac{\sqrt{3}}{2} - i\frac{1}{2} \implies \\ \left| \frac{\sqrt{3}}{2} - i\frac{1}{2} - 1+i \right| &= \left| \left(\frac{\sqrt{3}}{2} - 1 \right) + i\frac{1}{2} \right| \\ &= \sqrt{\left(\frac{\sqrt{3}}{2} - 1 \right)^2 + \left(\frac{1}{2} \right)^2} = \sqrt{2 - \sqrt{3}} \leq \frac{3}{2} \end{aligned}$$



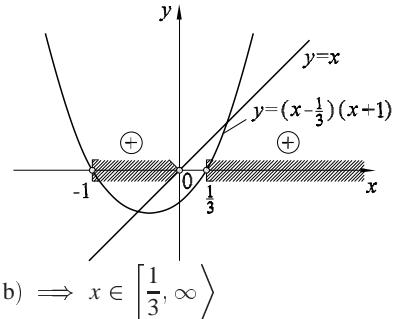
2. $f(x) = \sqrt{x+1 + \operatorname{sh}(\ln x)}$ je definirana za:

a) $x+1 + \operatorname{sh}(\ln x) \geq 0$
 $\iff x+1 + \frac{1}{2} (e^{\ln x} - e^{-\ln x}) \geq 0$

$$\begin{aligned} &\iff x+1 + \frac{1}{2} \left(x - \frac{1}{x} \right) \geq 0 \\ &\iff x \in [-1, 0) \cup \left[\frac{1}{3}, \infty \right) \end{aligned}$$

b) $\ln x$ je definiran za $x > 0$

$$\begin{aligned} &\iff \frac{3x^2 + 2x - 1}{2x} \geq 0 \\ &\iff \frac{3x^2 + 2x - 1}{x} \geq 0 \\ &\iff \frac{3(x - \frac{1}{3})(x + 1)}{x} \geq 0 \\ &\iff \frac{(x - \frac{1}{3})(x + 1)}{x} \geq 0 \\ &\iff x \in (0, \infty) \end{aligned}$$



a) \cap b) $\implies x \in \left[\frac{1}{3}, \infty \right)$

3. $\lim_{n \rightarrow \infty} \frac{1+5^n}{1+5+5^2+\dots+5^{n-1}+5^n} = \lim_{n \rightarrow \infty} \frac{1+5^n}{1-5^{n+1}} \overline{1-5}$
 $= 4 \lim_{n \rightarrow \infty} \frac{5^n + 1}{5^{n+1} - 1} = 4 \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{5^n}}{5 - \frac{1}{5^n}} = \frac{4}{5}$

4. a) Vrijedi

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2n + \sqrt{n}}{n} &= \lim_{n \rightarrow \infty} \left(2 + \frac{1}{\sqrt{n}} \right) = 2 \begin{cases} \neq 0, \\ \neq \infty \end{cases} \\ &\implies \sum_{n=1}^{\infty} \frac{1}{2n + \sqrt{n}} \sim \sum_{n=1}^{\infty} \frac{1}{n}. \end{aligned}$$

Kako $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira, zato i red $\sum_{n=1}^{\infty} \frac{1}{2n + \sqrt{n}}$ divergira.

b) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n n!}$. Primijenimo D'Alembertov kriterij:

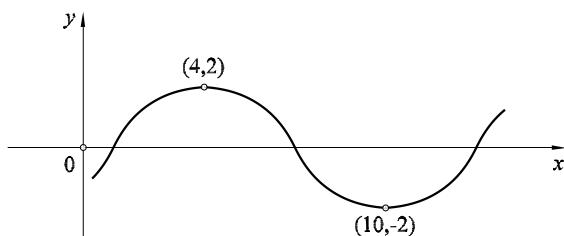
$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{[2(n+1)]!}{[2(n+1)]^{n+1}(n+1)!} \cdot \frac{(2n)^n n!}{(2n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(2n)!(2n+1)(2n+2)(2n)^n n!}{(2n+2)^{n+1} n!(n+1)(2n)!} \\ &= 2 \lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} \cdot \lim_{n \rightarrow \infty} \left(\frac{2n}{2n+2} \right)^n \\ &= 2 \underbrace{\lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2n}}{1 + \frac{2}{2n}}}_{=1} \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \\ &= 2 \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n} = \frac{2}{e} < 1 \end{aligned}$$

red konvergira.

Druga kontrolna zadaća

23. 02. 2001.

1. Naći jednadžbu sinusoide prikazane na slici i naći jednadžbu tangente na sinusoidu u točki u kojoj je $x = 1$.



Napomena. $(4, 2)$ i $(10, -2)$ su ekstremi.

2. Među svim pravokutnicima kojima dva vrha leže na osi x , a druga dva na krivulji $y = \sqrt{1 - 2x^2}$ (gornja polovica elipse), naći onaj koji ima maksimalnu površinu. Odrediti koordinate vrhova tog pravokutnika.
3. Odrediti područje definicije, ispitati ponašanje na rubu područja definicije, te naći asymptote, odrediti intervale monotonosti, te nacrtati kvalitativni graf funkcije

$$f(x) = \ln(2e^x - 1).$$

Primjedba. Ne traže se intervali konveksnosti i konkavnosti.

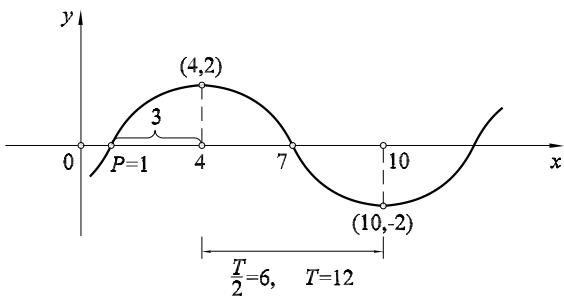
4. Izračunati integral:

$$\int_0^{16} \frac{dx}{(1 + \sqrt[4]{x})^2}.$$

Rješenja

23. 02. 2001.

1. $y = A \sin(\omega(x - p))$, $A = 2$, $T = 12$, $\omega = \frac{2\pi}{T} = \frac{\pi}{6}$,
 $p = 1$



Jednadžba sinusoide je

$$y = 2 \sin\left(\frac{\pi}{6}(x - 1)\right)$$

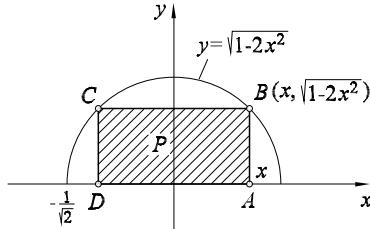
$$y'(x) = 2 \left[\cos\left(\frac{\pi}{6}(x - 1)\right) \right] \frac{\pi}{6}$$

$$y'(1) = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$$

Točka na sinusoidi s apscisom $x_0 = 1$ ima ordinatu $y_0 = 0$, pa je

$$y = \frac{\pi}{3}(x - 1) \quad \text{tangenta}$$

2. $P = 2x\sqrt{1-2x^2}$, $x \in \left[0, \frac{1}{\sqrt{2}}\right]$



$$P'(x) = 2 \left[\sqrt{1-2x^2} + x \frac{-4x}{2\sqrt{1-2x^2}} \right] = 2 \frac{1-4x^2}{\sqrt{1-2x^2}}$$

$$P'(x) = 0 \implies 1-4x^2 = 0 \implies x = \frac{1}{2} \quad (\text{zbog } x > 0).$$

Očito se radi o maksimumu

$$A\left(\frac{1}{2}, 0\right), B\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right), C\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right), D\left(-\frac{1}{2}, 0\right)$$

3. $f(x) = \ln(2e^x - 1)$ je definirana za $2e^x - 1 > 0 \iff e^x > \frac{1}{2} \iff x > \ln \frac{1}{2}$, $\mathcal{D}_f = \left(\ln \frac{1}{2}, +\infty\right)$

$$\lim_{x \rightarrow \ln \frac{1}{2}^+} f(x) = \lim_{x \rightarrow \ln \frac{1}{2}^+} \ln(2e^x - 1) = -\infty$$

$\implies x = \ln \frac{1}{2}$ vertikalna asimptota

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \ln(2e^x - 1) = +\infty$$

$$f'(x) = \frac{1}{2e^x - 1} \cdot 2e^x > 0 \text{ za } \forall x \in \mathcal{D}_f \implies$$

f strog rastuća \implies nema ekstrema.

Kosa asimptota: $y = kx + l$

$$k = \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \frac{\ln(2e^x - 1)}{x} = \left(\frac{\infty}{\infty}\right)$$

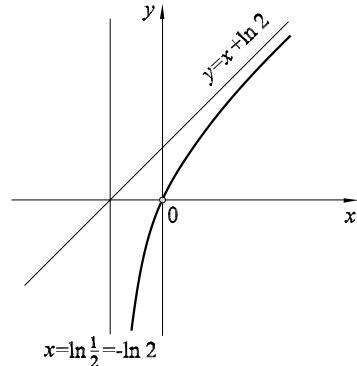
$$= \lim_{x \rightarrow +\infty} \frac{\frac{1}{2e^x - 1} \cdot 2e^x}{1} = \lim_{x \rightarrow +\infty} \frac{1}{1 - \frac{1}{2e^x}} = 1$$

$$l = \lim_{x \rightarrow +\infty} [f(x) - kx] = \lim_{x \rightarrow +\infty} [\ln(2e^x - 1) - x]$$

$$= \lim_{x \rightarrow +\infty} [\ln(2e^x - 1) - \ln e^x]$$

$$= \lim_{x \rightarrow +\infty} \ln\left(\frac{2e^x - 1}{e^x}\right) = \lim_{x \rightarrow +\infty} \ln\left(2 - \frac{1}{e^x}\right) = \ln 2$$

$\implies y = x + \ln 2$ desna kosa asimptota



4.

$$I = \int_0^{16} \frac{dx}{(1 + \sqrt[4]{x})^2} = \begin{vmatrix} 1 + \sqrt[4]{x} = t \implies x = (t-1)^4 \\ dx = 4(t-1)^3 dt \\ \text{za } x = 0 \rightarrow t = 1 \\ \text{za } x = 16 \rightarrow t = 3 \end{vmatrix}$$

$$= 4 \int_1^3 \frac{(t-1)^3}{t^2} dt = 4 \int_1^3 \left(t - 3 + \frac{3}{t} - \frac{1}{t^2} \right) dt$$

$$= \left. \left(2t^2 - 12t + 12 \ln t + \frac{4}{t} \right) \right|_1^3$$

$$= 16 - 24 + 12 \ln 3 - \frac{8}{3} = 12 \ln 3 - \frac{32}{3}$$

Pismeni ispit

30. 01. 2001.

1. Naći sve $z \in \mathbf{C}$ koji zadovoljavaju oba sljedeća uvjeta:

$$\begin{aligned} \arg(z^3) &= \frac{\pi}{2} \\ i \\ |z + 2| &= 1. \end{aligned}$$

2. Odrediti područje konvergencije reda i ispitati konvergenciju na rubu područja za red:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2\sqrt{n+1}}.$$

Sve tvrdnje detaljno obrazložiti.

3. U kružnicu zadanog polumjera r upisati trapez čija je dulja osnovica jednaka promjeru kružnice tako da mu je površina maksimalna. Koliko iznosi ta površina?

4. Neka je $f(x) = \frac{e^{\frac{2}{x}} + e^{\frac{1}{x}} + 2}{e^{\frac{2}{x}} + a}$.

a) Izračunati $\lim_{x \rightarrow 0+} f(x)$ i $\lim_{x \rightarrow 0-} f(x)$.

b) Za koju vrijednost parametra a postoji $\lim_{x \rightarrow 0} f(x)$?

5. Neka je $f(x) = \frac{\sqrt{x}}{\ln^2 x}$.

a) Odrediti domenu funkcije f , ispitati ponašanje na rubu područja definicije, te naći asimptote.

b) Odrediti intervale monotonosti i naći lokalne ekstrema.

c) Odrediti intervale konveksnosti i konkavnosti i naći točke infleksije.

d) Nacrtati kvalitativni graf funkcije.

6. Izračunati:

$$\int_1^{\sqrt{3}} \frac{x^5 + 1}{x^6 + x^4} dx.$$

7. Izračunati površinu lika omeđenog parabolom $y^2 = 4x$ i pravcem $y = 2x - 4$. Nacrtati sliku.

Rješenja

30. 01. 2001.

1. $\arg(z^3) = \frac{\pi}{2} \implies 3\arg z = \frac{\pi}{2} + 2k\pi$
 $\varphi = \arg z = \frac{\pi}{6} + \frac{2k\pi}{3}, k = 0, 1, 2.$

1. način.

I.

$$\varphi = \frac{\pi}{6}, z = r \operatorname{cis} \left(\frac{\pi}{6} \right) = r \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$$

$$|z + 2| = 1$$

$$\sqrt{\left(r \frac{\sqrt{3}}{2} + 2 \right)^2 + \frac{r^2}{4}} = 1$$

$$r^2 + 2\sqrt{3}r + 3 = 0$$

$$r = -\sqrt{3} < 0 \text{ nema rješenja}$$

Napomena. $[\operatorname{cis} \varphi = \cos \varphi + i \sin \varphi].$

II.

$$\varphi = \frac{5\pi}{6}, z = r \operatorname{cis} \left(\frac{5\pi}{6} \right) = r \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$$

$$|z + 2| = 1$$

$$\sqrt{\left(2 - r \frac{\sqrt{3}}{2} \right)^2 + \frac{r^2}{4}} = 1$$

$$r^2 - 2\sqrt{3}r + 3 = 0$$

$$r = \sqrt{3}$$

$$z = \sqrt{3} \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$$

III.

$$\varphi = \frac{3\pi}{2}, z = r \operatorname{cis} \left(\frac{3\pi}{2} \right) = -ri$$

$$|z + 2| = 1$$

$$\sqrt{4 + r^2} = 1$$

$$r^2 = -3 \text{ nema rješenja}$$

Dakle, jedino rješenje je $z = -\frac{3}{2} + \frac{\sqrt{3}}{2}i.$

2. način.

I. $\varphi = \frac{\pi}{6}, \frac{y}{x} = \frac{1}{\sqrt{3}}$; uvjet: $x > 0, y > 0$

$$|x + yi + 2| = 1$$

⋮

Nema rješenja koje zadovoljava uvjet.

II. $\varphi = \frac{5\pi}{6}, \frac{y}{x} = -\frac{1}{\sqrt{3}}$; uvjet: $x < 0, y > 0$

$$|x + yi + 2| = 1$$

⋮

$$x = -\frac{3}{2}, \quad y = \frac{\sqrt{3}}{2}$$

$$z = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$$

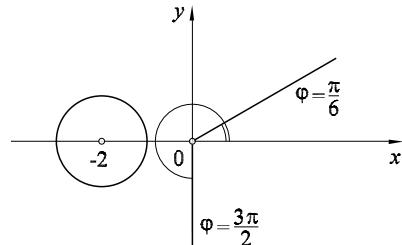
III. $\varphi = \frac{3\pi}{2}, x = 0$; uvjet: $y < 0$

$$|x + yi + 2| = 1$$

⋮

Nema rješenja koje zadovoljava uvjet.

3. način. Koristimo geometrijsku interpretaciju.



Očito nema rješenja za $\varphi = \frac{\pi}{6}$ i $\varphi = \frac{3\pi}{2}$ pa treba samo provjeriti $\varphi = \frac{5\pi}{6}$, što se može napraviti kao u 1. ili kao u 2. načinu.

2. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{2\sqrt{n+1}}$

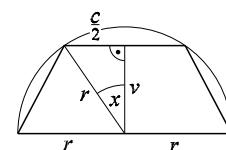
$$\lim_n \frac{|x-1|^{n+1}}{\frac{2\sqrt{n+1}+1}{2\sqrt{n+1}}} = |x-1| < 1, \quad x \in \langle 0, 2 \rangle$$

Kako je $\left\{ \frac{1}{2\sqrt{n+1}} \right\}$ padajući niz i $\lim_n \frac{1}{2\sqrt{n+1}} = 0$, po Leibnizovom kriteriju zaključujemo da red $\sum_n \frac{(-1)^n}{2\sqrt{n+1}}$ konvergira.

$$x = 2$$

$$\sum_n \frac{1}{2\sqrt{n+1}} \sim \frac{1}{2} \sum_n \frac{1}{\sqrt{n}} \text{ divergira}$$

3. 1. način.



$$\sin x = \frac{\frac{c}{2}}{r} \implies c = 2r \sin x$$

$$\cos x = \frac{v}{r} \implies v = r \cos x, \quad a = 2r$$

$$P = \frac{a+c}{2}v = \frac{2r(1+\sin x)}{2} \cdot r \cos x = r^2 \left(\cos x + \frac{1}{2} \sin 2x \right)$$

$$f(x) = \cos x + \frac{1}{2} \sin 2x, \quad x \in \left(0, \frac{\pi}{2} \right)$$

$$f'(x) = -\sin x + \frac{1}{2} \cos 2x \cdot 2 = -\sin x + 1 - 2 \sin^2 x$$

$$f'(x) = 0 \iff 2 \sin^2 x + \sin x - 1 = 0$$

$$\begin{aligned}\sin x &= \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} \\ \Rightarrow \sin x &= -1, \quad \sin x = \frac{1}{2}\end{aligned}$$

za $\sin x = -1 \Rightarrow x = \frac{3\pi}{2} + 2k\pi \notin \left(0, \frac{\pi}{2}\right)$

za $\sin x = \frac{1}{2} \Rightarrow x = \underbrace{\frac{\pi}{6}}_{\text{kritična točka}} \in \left(0, \frac{\pi}{2}\right)$

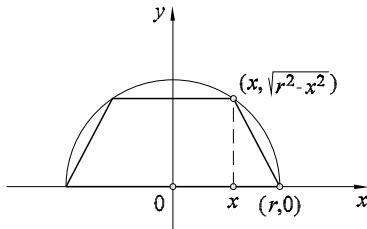
kritična točka

	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$
f'	+	-
f	/	\

$\Rightarrow x = \frac{\pi}{6}$ je točka maksimuma.

$$\begin{aligned}P\left(\frac{\pi}{6}\right) &= r^2 \left(\cos \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3}\right) = r^2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) \\ &= r^2 \frac{3\sqrt{3}}{4}\end{aligned}$$

2. način. (Koordinatizacija)



$$\begin{aligned}P &= (r+x)\sqrt{r^2-x^2}, \quad x \in \langle 0, r \rangle \\ P' &= \sqrt{r^2-x^2} + (r+x) \frac{(-2x)}{2\sqrt{r^2-x^2}} \\ &= \frac{r^2-x^2-rx-x^2}{\sqrt{r^2-x^2}} = \frac{-2x^2-rx+r^2}{\sqrt{r^2-x^2}} = 0 \\ x &= \frac{-r \pm \sqrt{3r}}{4}, \quad x = \frac{r}{2} \\ P_{\max} &= \frac{3r}{2} \cdot \frac{\sqrt{3}r}{2} = \frac{3\sqrt{3}}{4}r^2\end{aligned}$$

2. a) način. Bez formalne koordinatizacije imamo:

$$P = \left(r + \frac{c}{2}\right)v = \left(r + \frac{c}{2}\right)\sqrt{r^2 - \left(\frac{c}{2}\right)^2}$$

i dalje nastavljamo analogno.

4. a) $L_1 = \lim_{x \rightarrow 0+} \frac{e^{\frac{x}{2}} + e^{\frac{1}{x}} + 2}{e^{\frac{x}{2}} + a}$,

$$L_2 = \lim_{x \rightarrow 0-} \frac{e^{\frac{x}{2}} + e^{\frac{1}{x}} + 2}{e^{\frac{x}{2}} + a} = \frac{2}{a},$$

I. način za računanje L_1 (dijeljenje brojnika i nazivnika s $e^{\frac{2}{x}}$):

$$L_1 = \lim_{x \rightarrow 0+} \frac{1 + e^{-\frac{1}{x}} + 2e^{-\frac{2}{x}}}{1 + ae^{-\frac{2}{x}}} = 1$$

2. način za računanje L_1 (L'Hospital):

$$L_1 = \left(\frac{\infty}{\infty}\right) \stackrel{\text{L'H}}{\lim}_{x \rightarrow 0+} \frac{e^{\frac{x}{2}} \frac{(-2)}{x^2} + e^{\frac{1}{x}} \frac{(-1)}{x^2}}{e^{\frac{x}{2}} \frac{(-2)}{x^2}}$$

$$= \lim_{x \rightarrow 0+} \frac{2e^{\frac{x}{2}} + e^{\frac{1}{x}}}{2e^{\frac{x}{2}}} = \lim_{x \rightarrow 0+} 1 + \frac{1}{2}e^{-\frac{1}{x}} = 1$$

b) $\lim_{x \rightarrow 0} \frac{e^{\frac{x}{2}} + e^{\frac{1}{x}} + 2}{e^{\frac{x}{2}} + a}$ postoji ako i samo ako je $L_1 = L_2$, tj. $a = 2$.

5. $f(x) = y = \frac{\sqrt{x}}{\ln^2 x}$

1° $\mathcal{D}_f = \langle 0, 1 \rangle \cup \langle 1, +\infty \rangle$

2° nema nultočki, $f > 0$

3° $\lim_{x \rightarrow 0-} \frac{\sqrt{x}}{\ln^2 x} = 0, \lim_{x \rightarrow 1+} \frac{\sqrt{x}}{\ln^2 x} = +\infty, \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln^2 x} = +\infty$.

Pravac $x = 1$ je vertikalna asimptota.

4° $k = \lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{x \ln^2 x} = 0, l = \lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{\ln^2 x} \stackrel{\text{L'H}}{\dots} = +\infty$.

Nema kose niti horizontalne asimptote.

$$5° f'(x) = \frac{\frac{1}{2\sqrt{x}} \ln^2 x - 2\sqrt{x} \ln x \cdot \frac{1}{x}}{\ln^4 x} = \frac{\ln x - 4}{2\sqrt{x} \ln^3 x}, \mathcal{D}_{f'} = \mathcal{D}_f$$

$$f'(x) = 0 \iff \ln x - 4 \iff x = e^4$$

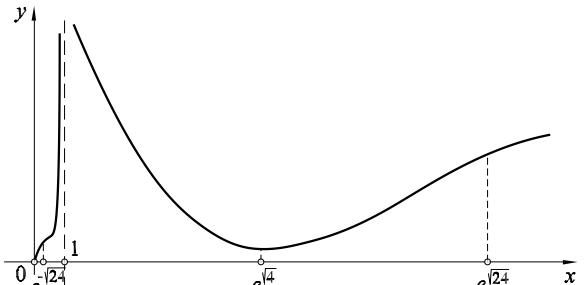
	$\langle 0, 1 \rangle$	$\langle 1, e^4 \rangle$	$\langle e^4, +\infty \rangle$
f'	+	-	+
f	/	\	/

lokalni
min

$$6° f''(x) = \dots = \frac{1}{4} \cdot \frac{24 - \ln^2 x}{\sqrt{x^3} \ln^4 x}, \mathcal{D}_{f''} = \mathcal{D}_f,$$

$$f''(x) = 0 \iff 24 - \ln^2 x = 0 \iff \begin{cases} x = e^{\sqrt{24}} \\ \text{ili} \\ x = \frac{1}{e^{\sqrt{24}}} \end{cases}$$

	$\langle 0, \frac{1}{e^{\sqrt{24}}} \rangle$	$\langle \frac{1}{e^{\sqrt{24}}}, 1 \rangle$	$\langle 1, e^{\sqrt{24}} \rangle$	$\langle e^{\sqrt{24}}, +\infty \rangle$
f''	-	+	+	-
f	/	\	\	/



6. $\frac{x^5 + 1}{x^4(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{Ex + F}{x^2 + 1}$

$$\begin{aligned}x^5 + 1 &= Ax^3(x^2 + 1) + Bx^2(x^2 + 1) + Cx(x^2 + 1) \\ &\quad + D(x^2 + 1) + x^4(Ex + F)\end{aligned}$$

$$\begin{aligned}
x^5 + 1 &= (A+E)x^5 + (B+F)x^4 + (A+C)x^3 \\
&\quad + (B+D)x^2 + Cx + D
\end{aligned}$$

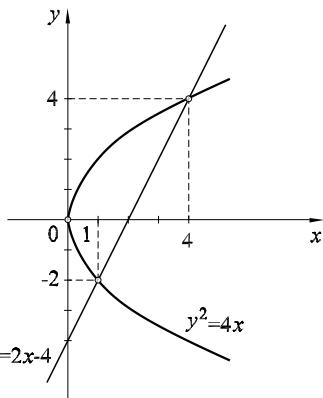
$$\begin{aligned}
D &= 1 & B+F &= 0 & A &= 0 \\
A+E &= 1 & A+C &= 0 \implies A &= 0 & B &= -1 \\
B+D &= 0 & & & C &= 0 \\
C &= 0 & & & D &= 1 \\
& & & & E &= 1 \\
& & & & F &= 1
\end{aligned}$$

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^5 + 1}{x^4(x^2 + 1)} dx &= \int_1^{\sqrt{3}} \left(-\frac{1}{x^2} + \frac{1}{x^4} + \frac{x+1}{x^2+1} \right) dx \\
&= \left(\frac{1}{x} - \frac{1}{3x^3} + \arctg x \right) \Big|_1^{\sqrt{3}} + \frac{1}{2} \underbrace{\int_1^{\sqrt{3}} \frac{2x}{x^2+1} dx}_{\ln(x^2+1)|_1^{\sqrt{3}}} \\
&= \frac{1}{\sqrt{3}} - 1 - \frac{1}{9\sqrt{3}} + \frac{1}{3} + \frac{\pi}{3} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) \\
&= \frac{8}{9\sqrt{3}} - \frac{2}{3} + \frac{\pi}{12} + \frac{1}{2} \ln 2
\end{aligned}$$

7.

$$\begin{aligned}
(2x-4)^2 &= 4x \\
4x^2 - 16x + 16 &= 4x \\
4x^2 - 20x + 16 &= 0 \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0
\end{aligned}$$

$$\begin{aligned}
x_1 &= 1, & x_2 &= 4 \\
y_1 &= -2, & y_2 &= 4
\end{aligned}$$



1. način.

$$\begin{aligned}
P &= \int_0^1 (2\sqrt{x} - (-2\sqrt{x})) dx + \int_1^4 (2\sqrt{x} - (2x-4)) dx \\
&= \int_0^1 4\sqrt{x} dx + \int_1^4 (2\sqrt{x} - 2x + 4) dx \\
&= 4 \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 + 2 \frac{2}{3} x^{\frac{3}{2}} \Big|_1^4 - 2 \frac{1}{2} x^2 \Big|_1^4 + 4x \Big|_1^4 = 9
\end{aligned}$$

2. način.

$$\begin{aligned}
P &= \int_{-2}^4 \left(\frac{y+4}{2} - \frac{y^2}{4} \right) dy = \frac{y^2}{4} \Big|_{-2}^4 + 2y \Big|_{-2}^4 - \frac{y^3}{12} \Big|_{-2}^4 \\
&= 3 + 12 - 6 = 9
\end{aligned}$$

Pismeni ispit

15. 02. 2001.

1. Naći sve $z \in \mathbf{C}$ koji zadovoljavaju jednadžbu:

$$z^5 + (1+i)^{10}z = 0.$$

2. Ispitati konvergenciju reda:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln\left(\frac{n+1}{n-1}\right).$$

3. Naći jednadžbu tangente na krivulju

$$xy^2 + x^3 + e^y = 2$$

u točki $(1, 0)$.

4. Odrediti pravokutni trokut minimalne površine čija hipotenuza leži na pravcu $y = x$, vrh nasuprot toj hipotenuzi leži na krivulji $y = \ln(x-1)$, dok su katete tog trokuta paralelne koordinatnim osima. Nacrtati sliku.

Primjedba. Treba odrediti koordinate vrhova tog trokuta.

5. Neka je $f(x) = \text{arc tg}\left(\frac{x^2}{x-1}\right)$.

- a) Odrediti domenu funkcije f , ispitati ponašanje na rubu područja definicije, te naći asimptote.
b) Odrediti intervale monotonosti i naći lokalne ekstreme.
c) Nacrtati kvalitativni graf funkcije.

6. Izračunati:

$$\int_0^1 \sqrt{x} e^{\sqrt{x}} dx.$$

7. Izračunati volumen rotacijskog tijela koje nastaje vrtnjom ograničenog lika omređenog krivuljom $y = \ln x$ i prvcima $x = 0$, $y = 0$, $y = -2$ oko osi y . Nacrtati sliku.

Rješenja

15. 02. 2001.

1. $z^5 + (1+i)^{10}z = 0 \iff z[z^4 + (1+i)^{10}] = 0$.

Imamo dvije mogućnosti:

a) $z_1 = 0$

ili

b)

$$\begin{aligned} z^4 &= -(1+i)^{10} = -\left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^{10} \\ &= -32 \operatorname{cis} \left(\frac{10\pi}{4}\right) = -32 \operatorname{cis} \left(\frac{\pi}{2}\right) = -32i \\ z_{2,3,4,5} &= \sqrt[4]{-32i} = \sqrt[4]{32 \operatorname{cis} \left(\frac{3\pi}{2}\right)} \\ &= \sqrt[4]{32} \operatorname{cis} \left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right), \quad k = 0, 1, 2, 3 \end{aligned}$$

2. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \ln \left(1 + \frac{2}{n-1}\right) \sim \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \cdot \frac{2}{n-1} \sim \sum_{n=2}^{\infty} \frac{2}{\sqrt{n} \cdot n}$
 $= 2 \sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ konvergira (Dirichletov red $p = \frac{3}{2} > 1$).

3. $xy^2 + x^3 + e^y = 2 \quad \frac{d}{dx}$
 $y^2 + 2xyy' + 3x^2 + e^y y' = 0$
 $3 + y'(1) = 0 \implies y'(1) = -3$

Jednadžba tangente je

$$y = -3(x-1) = -3x + 3$$

4. $P = P(x_0) = \frac{[x_0 - \ln(x_0 - 1)]^2}{2}, x_0 - 1 > 0, x_0 > 1$

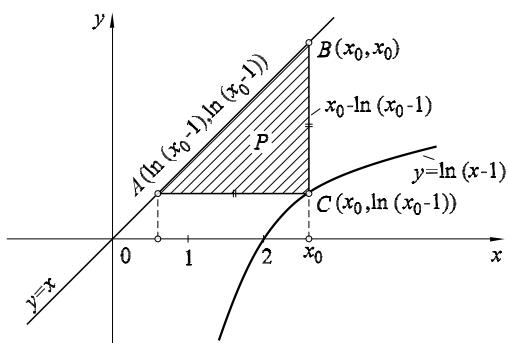
$P(x_0)$ minimum $\iff P_1(x_0) = x_0 - \ln(x_0 - 1)$ minimum

$$P'_1(x_0) = 1 - \frac{1}{x_0 - 1} = \frac{x_0 - 2}{x_0 - 1}$$

$$P'_1(x_0) = 0 \iff x_0 - 2 = 0 \iff x_0 = 2$$

$$P''_1(x_0) = \frac{1}{(x_0 - 1)^2} = 1 > 0 \implies \text{minimum za } x_0 = 2$$

Vrhovi trokuta su $A(0,0)$, $B(2,2)$, $C(2,0)$.



5. $f(x) = \arctg \left(\frac{x^2}{x-1} \right),$
 $\mathcal{D}_f = \mathbf{R} \setminus \{1\} = \langle -\infty, 1 \rangle \cup \langle 1, +\infty \rangle$

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \arctg \left(\lim_{x \rightarrow -\infty} \frac{x^2}{x-1} \right) = \begin{cases} x = -u \\ u \rightarrow +\infty \end{cases} \\ &= \arctg \left(\lim_{u \rightarrow +\infty} \frac{u^2}{-u-1} \right) = \arctg(-\infty) = -\frac{\pi}{2} \\ &\implies y = -\frac{\pi}{2} \text{ lijeva horizontalna asymptota} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} f(x) &= \dots = \frac{\pi}{2} \\ &\implies y = \frac{\pi}{2} \text{ desna horizontalna asymptota} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \arctg(-\infty) = -\frac{\pi}{2} \\ \lim_{x \rightarrow 1^+} f(x) &= \arctg(+\infty) = \frac{\pi}{2} \end{aligned}$$

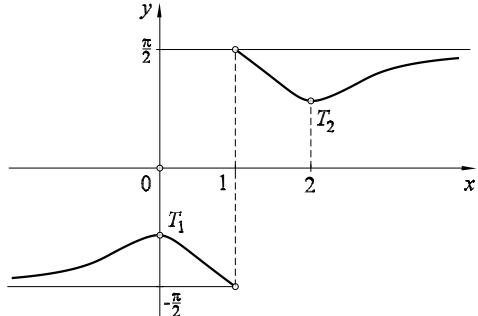
$$\begin{aligned} f'(x) &= \frac{1}{1 + \left(\frac{x^2}{x-1}\right)^2} \cdot \frac{2x(x-1) - x^2}{(x-1)^2} \\ &= \frac{1}{1 + \left(\frac{x^2}{x-1}\right)^2} \cdot \frac{x(x-2)}{(x-1)^2} \end{aligned}$$

$$f'(x) = 0 \implies x(x-2) = 0$$

$\implies x_1 = 0, x_2 = 2$ stacionarne točke

x	$\langle -\infty, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 1, 2 \rangle$	$\langle 2, +\infty \rangle$
$f'(x)$	+	-	-	+
$f(x)$	/	\	\	/

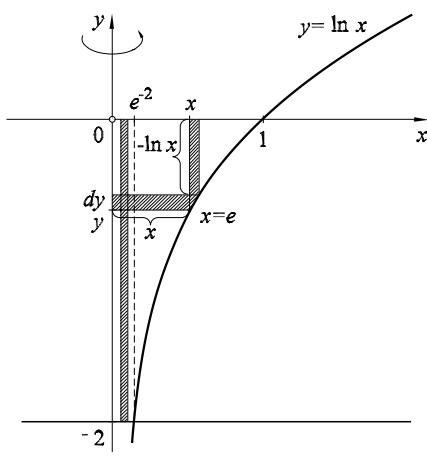
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6. $I = \int_0^1 \sqrt{x} e^{\sqrt{x}} dx = \begin{cases} \sqrt{x} = t^2, \quad dx = 2t dt \\ \text{za } x = 0 \implies t = 0 \\ \text{za } x = 1 \implies t = 1 \end{cases}$

$$\begin{aligned} &= 2 \int_0^1 t^2 e^t dt = 2 \int_0^1 \underbrace{t^2}_u d(\underbrace{e^t}_v) \\ &= 2 \left(t^2 e^t \Big|_0^1 - 2 \int_0^1 t e^t dt \right) = 2e - 4 \int_0^1 \underbrace{t}_u d(\underbrace{e^t}_v) \\ &= 2e - 4 \left(t e^t \Big|_0^1 - \int_0^1 e^t dt \right) = 2e - 4 \left(e - e^t \Big|_0^1 \right) \\ &= 2e - 4(e - e + 1) = 2e - 4 \end{aligned}$$

7. Za $y = -2 \Leftrightarrow \ln x = -2 \Leftrightarrow x = e^{-2}$, $y = \ln x \Leftrightarrow x = e^y$



1. nacín.

$$dV = x^2 \pi dy = (e^y)^2 \pi dy = \pi e^{2y} dy$$

$$V = \pi \int_{-2}^0 e^{2y} dy = \pi \frac{e^{2y}}{2} \Big|_{y=-2}^0$$

$$= \frac{\pi}{2} (1 - e^{-4})$$

2. nacín.

$$V = V_1 + V_2 = V_{\text{valjka}} + V_2$$

$$V_{\text{valjka}} = (e^{-2})^2 \pi \cdot 2 = 2\pi e^{-4}$$

$$V_2 = 2\pi \int_{e^{-2}}^1 x |y| dx = 2\pi \int_{e^{-2}}^1 x (-\ln x) dx$$

$$= \left| \begin{array}{l} u = -\ln x, du = -\frac{dx}{x} \\ dv = x dx \\ v = \int x dx = \frac{x^2}{2} \end{array} \right|$$

$$= 2\pi \left(-\frac{x^2}{2} \ln x \Big|_{e^{-2}}^1 + \int_{e^{-2}}^1 \frac{x^2}{2} \cdot \frac{dx}{x} \right)$$

$$= 2\pi \left(-e^{-4} + \frac{x^2}{4} \Big|_{e^{-2}}^1 \right)$$

$$= 2\pi \left(-e^{-4} + \frac{1}{4} - \frac{e^{-4}}{4} \right)$$

$$V = V_1 + V_2 = 2\pi e^{-4} + 2\pi \left(-e^{-4} + \frac{1}{4} - \frac{e^{-4}}{4} \right)$$

$$= \frac{\pi}{2} (1 - e^{-4})$$

Pismeni ispit

18. 04. 2001.

1. Naći sve $z \in \mathbf{C}$ koji zadovoljavaju jednadžbu:

$$z^6 + z^4 + z^2 + 1 = 0.$$

2. Niz $\{a_n\}$ je zadan na sljedeći način:

$$a_1 = 1, \quad a_{n+1} = \log_4(2^{1+a_n} + 8).$$

Dokazati da je niz $\{a_n\}$ konvergentan, te izračunati $\lim_{n \rightarrow \infty} a_n$.

3. Ispitati konvergenciju reda:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}.$$

4. Naći jednadžbu normale na krivulju

$$y = e^{2x} - e^x - 2$$

u točki u kojoj krivulja siječe os x .

5. Nacrtati kvalitativni graf funkcije

$$f(x) = \arcsin\left(\frac{1}{x}\right).$$

Primjedba. Ne treba tražiti intervale konveksnosti i konkavnosti.

6. Neka je $f(x) = \sin x \operatorname{sh} x$. Naći sve funkcije F takve da je $F' = f$.

7. Izračunati površinu lika određenog nejednadžbama:

$$\begin{aligned} x^2 + y^2 &\leq 2y, \\ y &\leq 2 - 2x^2. \end{aligned}$$

Nacrtati sliku!

Rješenja

18. 04. 2001.

$$\begin{aligned} \mathbf{1.} \quad z^6 + z^4 + z^2 + 1 = 0 &\iff z^4(z^2 + 1) + (z^2 + 1) = 0 \\ &\iff (z^2 + 1)(z^4 + 1) = 0 \end{aligned}$$

Imamo dvije mogućnosti:

a) $z^2 = -1 \implies z_{1,2} = \pm i$

ili

b) $z^4 = -1 = \text{cis } \pi$

$$\implies z = \sqrt[4]{\text{cis } \pi} = \text{cis} \left(\frac{\pi + 2k\pi}{4} \right), \quad k = 0, 1, 2, 3$$

$$z_{3,4,5,6} = \pm \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$$

$$\mathbf{2.} \quad a_1 = 1, a_{n+1} = \log_4(2^{1+a_n} + 8)$$

a) Dokaz da je niz rastući

$$a_1 = 1 < a_2 = \log_4 12$$

Pretp. $a_n < a_{n+1}$

$$2^{1+a_n} < 2^{1+a_{n+1}}$$

$$2^{1+a_n} + 8 < 2^{1+a_{n+1}} + 8$$

$$\log_4(2^{1+a_n} + 8) < \log_4(2^{1+a_{n+1}} + 8)$$

$$a_{n+1} < a_{n+2}$$

b) Dokaz da je niz $\{a_n\}$ omeđen odozgo

$$a_1 < 2$$

Pretp. $a_n < 2$

$$a_{n+1} = \log_4(2^{1+a_n} + 8) < \log_4(2^3 + 8) = \log_4 16 = 2$$

Iz a) i b) slijedi da je niz $\{a_n\}$ konvergentan, tj. postoji $\lim_{n \rightarrow \infty} a_n = L$

$$a_{n+1} = \log_4(2^{1+a_n} + 8) / \lim_{n \rightarrow \infty}$$

$$L = \log_4(2^{1+L} + 8)$$

$$4^L - 2 \cdot 2^L - 8 = 0$$

$$(2^L)^2 - 2 \cdot 2^L - 8 = 0$$

$$(2^L)_{1,2} = \frac{2 \pm \sqrt{4 + 32}}{2} = 1 \pm 3$$

$$\implies \begin{cases} 2^L = -2 & \text{otpada} \\ 2^L = 4 \implies L = 2 \end{cases}$$

$$\lim_{n \rightarrow \infty} a_n = 2$$

$$\mathbf{3.} \quad \sum_{n=1}^{\infty} \frac{\sin n}{n^2}, \quad \left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ konvergira} \implies \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \text{ konvergira}$$

$$\mathbf{4.} \quad y = e^{2x} - e^x - 2$$

$$y = 0 \implies e^{2x} - e^x - 2 = 0$$

$$(e^x)^2 - e^x - 2 = 0$$

$$e^x = \frac{1 \pm 3}{2} \implies \begin{cases} e^x = -1 & \text{otpada} \\ e^x = 2 \implies x = \ln 2 \end{cases}$$

$$T(\ln 2, 0)$$

$$y'(x) = 2e^{2x} - e^x$$

$$y'(\ln 2) = 2e^{2 \ln 2} - e^{\ln 2}$$

$$y'(\ln 2) = 2e^{\ln 4} - e^{\ln 2}$$

$$= 2 \cdot 4 - 2 = 8 - 2 = 6$$

Normala:

$$n \dots y - 0 = -\frac{1}{6}(x - \ln 2)$$

$$y = -\frac{1}{6}(x - \ln 2)$$

$$\mathbf{5.} \quad f(x) = \arcsin\left(\frac{1}{x}\right) \text{ definirana za } \begin{cases} \left|\frac{1}{x}\right| \leq 1 \\ x \neq 0 \end{cases} \implies$$

$$|x| \geq 1, \quad \mathcal{D}_f = (-\infty, -1] \cup [1, +\infty)$$

$$\lim_{x \rightarrow -\infty} f(x) = \arcsin 0 = 0$$

$$\lim_{x \rightarrow +\infty} f(x) = \arcsin 0 = 0$$

$\implies y = 0$ horizontalna asimptota

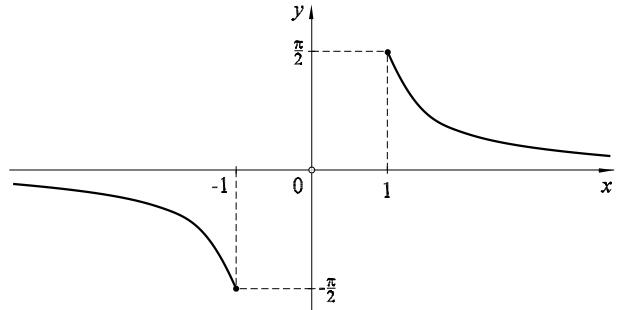
$$\lim_{x \rightarrow -1^-} f(x) = \arcsin(-1) = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow 1^+} f(x) = \arcsin 1 = \frac{\pi}{2}$$

Ekstremi:

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \left(-\frac{1}{x^2} \right) \neq 0, \quad \forall x \in \mathcal{D}_f$$

\implies nema ekstrema



6.

$$I = \int \sin x \operatorname{sh} x dx = \int \underbrace{\sin x}_u d(\underbrace{\operatorname{ch} x}_v)$$

$$= \sin x \operatorname{ch} x - \int \cos x \underbrace{\operatorname{ch} x dx}_{d(\operatorname{sh} x)}$$

$$= \sin x \operatorname{ch} x - \left(\cos x \operatorname{sh} x + \underbrace{\int \sin x \operatorname{sh} x dx}_{=I} \right)$$

$$I = \int \sin x \operatorname{sh} x dx$$

$$= \frac{1}{2} \sin x \operatorname{ch} x - \frac{1}{2} \cos x \operatorname{sh} x + C$$

$$7. \quad x^2 + y^2 \leq 2y \iff x^2 + (y-1)^2 \leq 1,$$

$$y \leq 2 - 2x^2$$

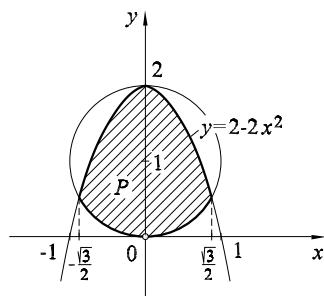
$$x^2 + y^2 = 2y$$

$$\underline{y = 2 - 2x^2}$$

$$x^2 + (2 - 2x^2) = 2(2 - 2x^2)$$

$$4x^4 - 3x^2 = 0$$

$$x^2(4x^2 - 3) = 0 \implies x_1 = 0, \quad x_{2,3} = \pm \frac{\sqrt{3}}{2}$$



$$P = 2 \left[\int_0^{\frac{\sqrt{3}}{2}} (2 - 2x^2) dx - \int_0^{\frac{\sqrt{3}}{2}} (1 - \sqrt{1 - x^2}) dx \right]$$

$$= \frac{\sqrt{3}}{2} + 2 \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1 - x^2} dx$$

$$= \begin{cases} x = \sin t, \quad dx = \cos t dt \\ x = 0 \implies t = 0 \\ x = \frac{\sqrt{3}}{2} \implies t = \frac{\pi}{3} \end{cases}$$

$$= \frac{\sqrt{3}}{2} + 2 \int_0^{\frac{\pi}{3}} \cos^2 t dt = \frac{\sqrt{3}}{2} + 2 \int_0^{\frac{\pi}{3}} (1 + \cos 2t) dt$$

$$= \frac{\sqrt{3}}{2} + \left(t + \frac{\sin 2t}{2} \right) \Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{1}{2} \sin\left(\frac{2\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{4} + \frac{\pi}{3}$$

Pismeni ispit

19. 06. 2001.

1. Naći sve $z \in \mathbf{C}$ koji zadovoljavaju oba sljedeća uvjeta:

$$|z + 2 + i| = 1, \quad |z + 2 - i| = 3.$$

2. Naći područje konvergencije reda i ispitati konvergenciju na rubu područja za red:

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n+1} + \sqrt{n}}.$$

Obrazložiti!

3. Naći drugi Taylorov polinom $T_2(x)$ u razvoju funkcije $y = y(x)$ koja je zadana parametarski:

$$x(t) = t^2 - 3t + 2,$$

$$y(t) = t^2 - 5t + 6$$

oko točke $(0, 0)$.

4. Naći lijevu i desnu kosu asymptotu krivulje:

$$y = 2x + 1 - \sqrt{x^2 - x + 1}.$$

5. Odrediti područje definicije, ispitati ponašanje na rubu područja definicije, naći lokalne ekstreme i asymptote, te nacrtati kvalitativni graf funkcije:

$$f(x) = e^{\frac{1}{x^2-x-2}}.$$

6. Izračunati:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\cos x \cdot \sin^2 x}.$$

7. Sa $[x]$ označimo najveći cijeli broj manji ili jednak $x \in \mathbf{R}$ (npr. $[2] = 2$, $[2.2] = 2$, $[1.9] = 1$). Izračunati:

$$\int_1^{\frac{5}{2}} [x] dx.$$

Rješenja

19. 06. 2001.

$$1. \begin{cases} |z+2+i|=1 \\ |z+2-i|=3 \end{cases}$$

$$\begin{cases} |(x+2)+(y+1)i|=1 \\ |(x+2)+(y-1)i|=3 \end{cases}$$

$$\begin{cases} (x+2)^2+(y+1)^2=1 \\ (x+2)^2+(y-1)^2=3^2 \end{cases}$$

$$\begin{cases} (x+2)^2=1-(y+1)^2 \\ (x+2)^2=9-(y-1)^2 \end{cases}$$

$$1-(y+1)^2=9-(y-1)^2$$

$$1-y^2-2y-1=9-y^2+2y-1$$

$$4y+8=0 \implies y=-2$$

$$(x+2)^2=1-(-2+1)^2$$

$$(x+2)^2=0 \implies x=-2$$

$$\implies z=-2-2i$$

$$2. \sum_{n=1}^{\infty} \underbrace{\frac{(x-1)^n}{\sqrt{n+1}+\sqrt{n}}}_{a_n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{n+1}}{\sqrt{n+2}+\sqrt{n+1}}}{\frac{(x-1)^n}{\sqrt{n+1}+\sqrt{n}}} \right|$$

$$= |x-1| \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} = |x-1|$$

Red konvergira za $|x-1| < 1 \iff -1 < x-1 < 1 \iff 0 < x < 2$.

Za $x=0$ red $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ konvergira po Leibnizovu kriteriju.

Za $x=2$ red $\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ divergira.

Rješenje. $0 \leq x < 2$ ili $x \in [0, 2)$

$$3. \begin{cases} x(t) = t^2 - 3t + 2 \\ y(t) = t^2 - 5t + 6 \end{cases}$$

$T(0, 0)$ I. $0 = t^2 - 3t + 2 \implies t_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$

$$\implies \begin{cases} t_1 = 1 & \text{ne zadovoljava} \\ t_2 = 2 & \end{cases}$$

II. $0 = t^2 - 5t + 6 \implies$

za $t_1 = 1 \implies 0 = 1 - 5 + 6 \implies 0 \neq 2$

za $t_2 = 2 \implies 0 = 4 - 10 + 6 \implies 0 = 0$

$t_2 = 2$ zadovoljava obje jednadžbe. Točki $T(0, 0)$ odgovara parametar $t_2 = 2$.

$$y'(x) = \frac{dy}{dx} = \frac{\frac{dt}{dx}}{\frac{dt}{dx}} = \frac{2t-5}{2t-3}$$

$$y'(0) = \left(\frac{2t-5}{2t-3} \right)_{t=2} = \frac{4-5}{4-3} = \frac{-1}{1} = -1$$

$$y''(x) = \frac{dy'(x)}{dx} = \frac{d}{dx} \left(\frac{2t-5}{2t-3} \right) = \frac{\frac{d}{dt} \left(\frac{2t-5}{2t-3} \right)}{\frac{dx}{dt}}$$

$$= \frac{1}{2t-3} \cdot \frac{2(t-3)-(2t-5) \cdot 2}{(2t-3)^2} = \frac{4}{(2t-3)^3}$$

$$y''(0) = \left[\frac{4}{(2t-3)^3} \right]_{t=2} = \frac{4}{1^3} = 4$$

$$T_2(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2$$

$$T_2(x) = 0 - \frac{1}{1!}x + \frac{4}{2!}x^2 = 2x^2 - x$$

4. $y = 2x + 1 - \sqrt{x^2 - x + 1}$

Desna kosa asimptota: $y = k_1 x + l_1$

$$k_1 = \lim_{x \rightarrow \infty} \frac{y}{x} = \lim_{x \rightarrow \infty} \frac{2x+1-\sqrt{x^2-x+1}}{x}$$

$$= \lim_{x \rightarrow \infty} \left(2 + \frac{1}{x} - \sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} \right) = 2 - 1 = 1,$$

$$l_1 = \lim_{x \rightarrow \infty} (y - k_1 x) = \lim_{x \rightarrow \infty} (2x+1-\sqrt{x^2-x+1}-x)$$

$$= \lim_{x \rightarrow \infty} (x+1-\sqrt{x^2-x+1})$$

$$= 1 + \lim_{x \rightarrow \infty} (x-\sqrt{x^2-x+1})$$

$$= 1 + \lim_{x \rightarrow \infty} \frac{(x-\sqrt{x^2-x+1})(x+\sqrt{x^2-x+1})}{x+\sqrt{x^2-x+1}}$$

$$= 1 + \lim_{x \rightarrow \infty} \frac{x^2-x^2+x-1}{x+\sqrt{x^2-x+1}}$$

$$= 1 + \lim_{x \rightarrow \infty} \frac{1-\frac{1}{x}}{1+\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}} = 1 + \frac{1}{1+1} = \frac{3}{2}$$

$\implies y = x + \frac{3}{2}$ desna kosa asimptota

Lijeva kosa asimptota: $y = k_2 x + l_2$

$$k_2 = \lim_{x \rightarrow -\infty} \frac{y}{x} = \lim_{x \rightarrow -\infty} \frac{2x+1-\sqrt{x^2-x+1}}{x}$$

$$= \begin{vmatrix} x = -t \\ x \rightarrow -\infty \\ t \rightarrow +\infty \end{vmatrix} = \lim_{t \rightarrow +\infty} \frac{-2t+1-\sqrt{t^2+t+1}}{-t}$$

$$= \lim_{t \rightarrow +\infty} \frac{2t-1+\sqrt{t^2+t+1}}{t}$$

$$= \lim_{t \rightarrow +\infty} \left(2 - \frac{1}{t} + \sqrt{1 + \frac{1}{t} + \frac{1}{t^2}} \right) = 2 + 1 = 3$$

$$l_2 = \lim_{x \rightarrow -\infty} (y - k_2 x)$$

$$= \lim_{x \rightarrow -\infty} (2x+1-\sqrt{x^2-x+1}-3x)$$

$$= \lim_{x \rightarrow -\infty} (-x+1-\sqrt{x^2-x+1})$$

$$= \begin{vmatrix} x = -t \\ x \rightarrow -\infty \\ t \rightarrow +\infty \end{vmatrix} = \lim_{t \rightarrow +\infty} (t+1-\sqrt{t^2+t+1})$$

$$= 1 + \lim_{t \rightarrow +\infty} \frac{(t-\sqrt{t^2+t+1})(t+\sqrt{t^2+t+1})}{t+\sqrt{t^2+t+1}}$$

$$\begin{aligned}
&= 1 + \lim_{t \rightarrow +\infty} \frac{t^2 - t^2 - t - 1}{t + \sqrt{t^2 + t + 1}} \\
&= 1 + \lim_{t \rightarrow +\infty} \frac{-1 - \frac{1}{t}}{1 + \sqrt{1 + \frac{1}{t} + \frac{1}{t^2}}} = 1 - \frac{1}{2} = \frac{1}{2} \\
\implies y &= 3x + \frac{1}{2} \text{ lijeva kosa asimptota}
\end{aligned}$$

5. $f(x) = e^{\frac{1}{x^2-x-2}}$ definirano za $x^2 - x - 2 \neq 0$;

$$x^2 - x - 2 = 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}, \quad x_1 = -1, \quad x_2 = 2;$$

$$\mathcal{D}_f = \mathbf{R} \setminus \{-1, 2\} = \langle -\infty, -1 \rangle \cup \langle -1, 2 \rangle \cup \langle 2, +\infty \rangle;$$

$$\lim_{x \rightarrow \pm\infty} f(x) = e^{\lim_{x \rightarrow -\infty} \frac{1}{(x+1)(x-2)}} = e^0 = 1$$

$$\implies y = 1 \text{ horizontalna asimptota}$$

$$\lim_{x \rightarrow -1^-} f(x) = e^{\lim_{x \rightarrow -1^-} \frac{1}{(x+1)(x-2)}} = e^{\lim_{x \rightarrow -1^-} \frac{1}{x-2} \cdot \lim_{x \rightarrow -1^-} \frac{1}{x+1}} = e^{-\frac{1}{3} \lim_{x \rightarrow -1^-} \frac{1}{x+1}} = e^{+\infty} = +\infty$$

$$\lim_{x \rightarrow -1^+} f(x) = \dots = e^{-\frac{1}{3} \lim_{x \rightarrow -1^+} \frac{1}{x+1}} = e^{-\infty} = 0$$

$x = -1$ vertikalna asimptota

$$\begin{aligned}
\lim_{x \rightarrow 2^-} f(x) &= e^{\lim_{x \rightarrow 2^-} \frac{1}{x+1} \cdot \lim_{x \rightarrow 2^-} \frac{1}{x-2}} = e^{\frac{1}{3} \lim_{x \rightarrow 2^-} \frac{1}{x-2}} = e^{-\infty} = 0 \\
\lim_{x \rightarrow 2^+} f(x) &= \dots = e^{\frac{1}{3} \lim_{x \rightarrow 2^+} \frac{1}{x-2}} = e^{+\infty} = +\infty
\end{aligned}$$

$x = 2$ vertikalna asimptota

Ekstremi:

$$f'(x) = e^{\frac{1}{x^2-x-2}} \frac{-(2x-1)}{(x^2-x-2)^2},$$

$$f'(x) = 0 \implies 2x - 1 = 0, \quad x = \frac{1}{2} \text{ stacionarna točka}$$

$$y_{\max} = f\left(\frac{1}{2}\right) = e^{-\frac{4}{9}} = \frac{1}{\sqrt[9]{e^4}}$$

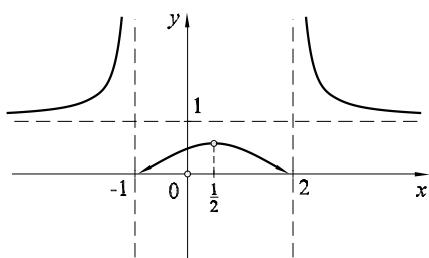
x	$\langle -\infty, -1 \rangle$	-1	$\langle -1, \frac{1}{2} \rangle$	$\frac{1}{2}$	$\langle \frac{1}{2}, 2 \rangle$	2	$\langle 2, +\infty \rangle$
$f'(x)$	+	nije def.	+	0	-	nije def.	-
$f(x)$	/	def.	/	max	\	def.	\

Asimptote:

vertikalne: $x = 1$ i $x = 2$

horizontalna: $y = 1$

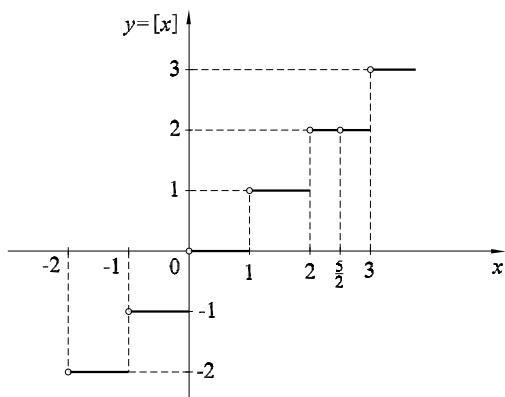
kosih nema



6.

$$\begin{aligned}
I &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\cos x \sin^2 x} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x \, dx}{\cos^2 x \sin^2 x} \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d(\sin x)}{(1 - \sin^2 x) \sin^2 x} = \left| \begin{array}{l} \sin x = t \\ x = \frac{\pi}{4} \implies t = \frac{\sqrt{2}}{2} \\ x = \frac{\pi}{3} \implies t = \frac{\sqrt{3}}{2} \end{array} \right| \\
&= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dt}{(1 - t^2)t^2} = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dt}{(t^2 - 1)t^2} = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dt}{t^2(t-1)(t+1)} \\
&= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1} + \frac{D}{t+1} \right) dt \\
&= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \left(-\frac{1}{t^2} + \frac{1}{2} \cdot \frac{1}{t-1} - \frac{1}{2} \cdot \frac{1}{t+1} \right) dt \\
&= \left(\frac{1}{t} + \frac{1}{2} \ln |t-1| - \frac{1}{2} \ln |t+1| \right) \Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \\
&= \left(\frac{1}{t} + \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \right) \Big|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \\
&= \frac{2}{\sqrt{2}} + \frac{1}{2} \ln \left| \frac{\frac{\sqrt{2}}{2}-1}{\frac{\sqrt{2}}{2}+1} \right| - \frac{2}{\sqrt{3}} - \frac{1}{2} \ln \left| \frac{\frac{\sqrt{3}}{2}-1}{\frac{\sqrt{3}}{2}+1} \right| \\
&= \sqrt{2} + \frac{1}{2} \ln \left| \frac{\sqrt{2}-2}{\sqrt{2}+2} \right| - \frac{2}{3}\sqrt{3} - \frac{1}{2} \ln \left| \frac{\sqrt{3}-2}{\sqrt{3}+2} \right| \\
&= \frac{3\sqrt{2}-2\sqrt{3}}{3} + \frac{1}{2} \ln \left| \frac{(\sqrt{2}-2)(\sqrt{3}+2)}{(\sqrt{2}+2)(\sqrt{3}-2)} \right|
\end{aligned}$$

7. $I = \int_{-2}^{\frac{5}{2}} [x] \, dx = \int_1^2 1 \, dx + \int_2^{\frac{5}{2}} 2 \, dx = x \Big|_1^2 + 2x \Big|_2^{\frac{5}{2}} = 2 - 1 + 2 \cdot \frac{5}{2} - 4 = 2$



Pismeni ispit

27. 06. 2001.

1. Odrediti polinom 3. stupnja P s realnim koeficijentima tako da vrijedi $P(1 - i) = 2 + i$ i $P(i) = 1 - 2i$.

2. Izračunati sumu reda:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

3. Dva vrha pravokutnika nalaze se na krivulji $y = \frac{x^2 + 2}{x^2 + 1}$, a druga dva na krivulji $y = \frac{x^2}{x^2 + 1}$. Odrediti vrhove pravokutnika tako da njegova površina bude maksimalna. Nacrtati sliku!

4. Izračunati:

$$\lim_{x \rightarrow \frac{\pi}{2}} ((1 - \sin x) \cdot \operatorname{tg} x).$$

5. Odrediti područje definicije, ispitati ponašanje na rubu područja definicije, naći intervale monotonosti, lokalne ekstreme i asimptote, te nacrtati kvalitativni graf funkcije:

$$f(x) = \frac{1}{e^{2x} - 2e^x + 2}.$$

Primjedba. Ne treba tražiti intervale konveksnosti i konkavnosti.

6. Izračunati:

$$\int \frac{\ln x \, dx}{x \cdot \sqrt{1 - 4 \ln x - \ln^2 x}}.$$

7. Izračunati površinu lika omeđenog krivuljom $y = \ln\left(x^2 + \frac{3}{4}e^2\right)$ i pravcem $y = 2$. Nacrtati sliku!

Rješenja

27. 06. 2001.

1. $P(z) = az^3 + bz^2 + cz + d, a, b, c, d \in \mathbf{R}$

$$\begin{aligned} P(1-i) &= a(1-i)^3 + b(1-i)^2 + c(1-i) + d \\ &= -2a + c + d + i(-2a - 2b - c) = 2 + i \\ P(i) &= ai^3 + bi^2 + ci + d \\ &= -b + d + i(-a + c) = 1 - 2i \\ \begin{cases} -2a + c + d = 2 \\ -2a - 2b - c = 1 \end{cases} \\ \begin{cases} -b + d = 1 \implies d = b + 1 \\ -a + c = -2 \implies c = a - 2 \end{cases} \\ \begin{cases} -a + b = 3 \\ -3a - 2b = -1 \end{cases} \\ a = -1, b = 2, c = -3, d = 3 \end{aligned}$$

Rješenje:

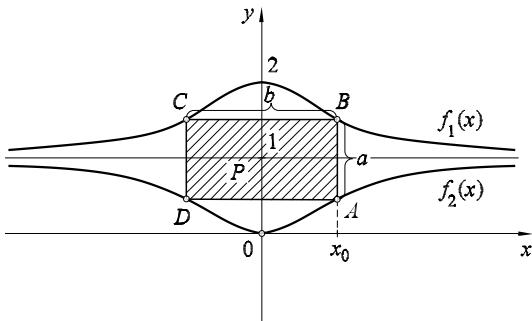
$$P(z) = -z^3 + 2z^2 - 3z + 3$$

2.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n(n+2)} &= \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \right] \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right] \\ &= \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4} \end{aligned}$$

3.

$$\begin{aligned} f_1(x) &= \frac{x^2 + 2}{x^2 + 1} = 1 + \frac{1}{x^2 + 1} \\ f_2(x) &= \frac{x^2}{x^2 + 1} = 1 - \frac{1}{x^2 + 1} \end{aligned}$$



$$P = ab$$

$$\begin{aligned} a &= f_1(x_0) - f_2(x_0) = 1 + \frac{1}{x_0^2 + 1} - 1 + \frac{1}{x_0^2 + 1} \\ &= \frac{2}{x_0^2 + 1} \end{aligned}$$

$$b = 2x_0$$

$$P(x_0) = \frac{4x_0}{x_0^2 + 1}, \quad x_0 > 0$$

$$P'(x_0) = 4 \frac{x_0^2 + 1 - x_0 \cdot 2x_0}{(x_0^2 + 1)^2} = 4 \frac{1 - x_0^2}{(x_0^2 + 1)^2}$$

$$P'(x_0) = 0 \iff 1 - x_0^2 = 0$$

Zbog $x_0 > 0 \implies x_0 = 1$. Očito se radi o maksimumu.
Vrhovi pravokutnika su:

$$A\left(1, \frac{1}{2}\right), B\left(1, \frac{3}{2}\right), C\left(-1, \frac{3}{2}\right), D\left(-1, \frac{1}{2}\right).$$

4.

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}} [(1 - \sin x) \operatorname{tg} x] &= (0 \cdot \infty) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\operatorname{ctg} x} \\ &= \left(\frac{0}{0}\right) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cos x}{-\frac{1}{\sin^2 x}} = 0 \end{aligned}$$

5. $f(x) = \frac{1}{e^{2x} - 2e^x + 2}$ definirana za $e^{2x} - 2e^x + 2 \neq 0$;
 $(e^x)^2 - 2e^x + 2 = 0, (e^x)_{1,2} = 1 \pm i \notin \mathbf{R} \implies \mathcal{D}(f) = \mathbf{R}$

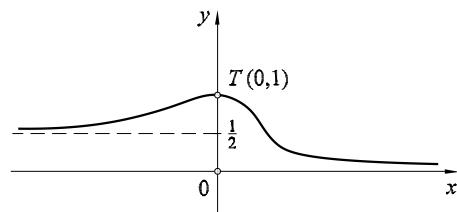
$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{1}{e^{2x} - 2e^x + 2} = \frac{1}{2} \\ \implies y = \frac{1}{2} &\text{ lijeva horizontalna asimptota} \\ \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} \frac{1}{e^{2x} - 2e^x + 2} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{e^{2x}} \cdot \frac{1}{1 - 2e^{-x} + 2e^{-2x}} = 0 \\ \implies y = 0 &\text{ desna horizontalna asimptota} \end{aligned}$$

Ekstremi:

$$\begin{aligned} f'(x) &= \frac{-(2e^{2x} - 2e^x)}{(e^{2x} - 2e^x + 2)^2} = \frac{2e^x(1 - e^x)}{(e^{2x} - 2e^x + 2)^2} \\ f'(x) = 0 &\implies 1 - e^x = 0 \implies x = \ln 1 = 0 \end{aligned}$$

x	$\langle -\infty, 0 \rangle$	0	$\langle 0, +\infty \rangle$
$f'(x)$	+	0	-
$f(x)$	/	max	/

$T(0, 1)$ – točka maksimuma



6.

$$\begin{aligned} I &= \int \frac{\ln x \, dx}{x \sqrt{1 - 4 \ln x - \ln^2 x}} = \left| \begin{array}{l} \ln x = t / d \\ \frac{dx}{x} = dt \end{array} \right| \\ &= \int \frac{t \, dt}{\sqrt{1 - 4t - t^2}} = \int \frac{t \, dt}{\sqrt{5 - (t+2)^2}} \\ &= \int \frac{(t+2) \, d(t+2)}{\sqrt{5 - (t+2)^2}} - 2 \int \frac{d(t+2)}{\sqrt{5 - (t+2)^2}} \end{aligned}$$

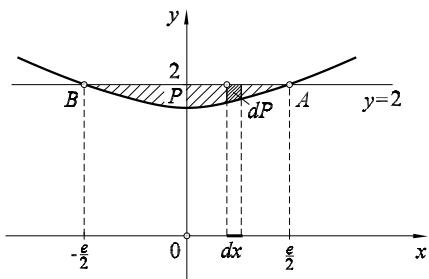
$$\begin{aligned}
&= - \int d \left(\sqrt{5 - (t+2)^2} \right) - 2 \arcsin \left(\frac{t+2}{\sqrt{5}} \right) + C \\
&= -\sqrt{5 - (t+2)^2} - 2 \arcsin \left(\frac{t+2}{\sqrt{5}} \right) + C \\
&= -\sqrt{5 - (\ln x + 2)^2} - 2 \arcsin \left(\frac{\ln x + 2}{\sqrt{5}} \right) + C \\
&= -\sqrt{1 - 4 \ln x - \ln^2 x} - 2 \arcsin \left(\frac{\ln x + 2}{\sqrt{5}} \right) + C
\end{aligned}$$

7.

$$\begin{aligned}
\ln \left(x^2 + \frac{3}{4} e^2 \right) &= 2 \\
x^2 + \frac{3}{4} e^2 &= e^2 \iff x^2 = \frac{e^2}{4} \iff x = \pm \frac{e}{2}
\end{aligned}$$

$$P = \frac{1}{2} \int_0^{\frac{e}{2}} \left[2 - \ln \left(x^2 + \frac{3}{4} e^2 \right) \right] dx$$

$$P = 2e - 2 \int_0^{\frac{e}{2}} \ln \left(x^2 + \frac{3}{4} e^2 \right) dx = 2e - 2I$$



$$\begin{aligned}
I &= \int_0^{\frac{e}{2}} \underbrace{\ln \left(x^2 + \frac{3}{4} e^2 \right)}_u \underbrace{dx}_v \\
&= x \ln \left(x^2 + \frac{3}{4} e^2 \right) \Big|_{x=0}^{\frac{e}{2}} - \int_0^{\frac{e}{2}} x \frac{2x}{x^2 + \frac{3}{4} e^2} dx \\
&= e - 2 \int_0^{\frac{e}{2}} \frac{\left(x^2 + \frac{3}{4} e^2 \right) - \frac{3}{4} e^2}{x^2 + \frac{3}{4} e^2} dx \\
&= e - 2 \int_0^{\frac{e}{2}} dx + \frac{3}{2} e^2 \int_0^{\frac{e}{2}} \frac{dx}{x^2 + \left(\frac{e}{2} \sqrt{3} \right)^2} \\
&= e - 2 \frac{e}{2} + \frac{3e^2}{2} \cdot \frac{2}{e\sqrt{3}} \operatorname{arc tg} \left(\frac{2x}{e\sqrt{3}} \right) \Big|_{x=0}^{\frac{e}{2}} \\
&= e\sqrt{3} \operatorname{arc tg} \left(\frac{2}{e\sqrt{3}} \cdot \frac{e}{2} \right) \\
&= e\sqrt{3} \operatorname{arc tg} \left(\frac{1}{\sqrt{3}} \right) \\
&= e\sqrt{3} \frac{\pi}{6}
\end{aligned}$$

Rješenje:

$$\begin{aligned}
P &= 2e - 2I = 2e - 2e\sqrt{3} \frac{\pi}{6} \\
&= 2e \left(1 - \frac{\sqrt{3}}{6} \pi \right)
\end{aligned}$$