Chapter 1

Euler integral identities

1.1 Introduction

Integral Euler identities extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials (cf. [79] or Appendix) and were derived in [30]. To prove them, the following lemma is needed:

Lemma 1.1 *Let $a,b \in \mathbb{R}$, $a < b$, $x \in [a,b]$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by*

$$\varphi(t) = B_1^*\left(\frac{x-t}{b-a}\right).$$

Then for every continuous function $F : [a,b] \to \mathbb{R}$ we have

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(x),\ \textit{for}\ \ a < x < b$$

and

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(a),\ \textit{for}\ \ x = a\ \ \textit{or}\ \ x = b,$$

with Riemann-Stieltjes integrals on the left hand sides.

Proof. If $a < x < b$ the function φ is differentiable on $[a,b]\backslash\{x\}$ and its derivative is equal to $\frac{-1}{b-a}$, since $B_1(t) = t - 1/2$. It has a jump of $\varphi(x+0) - \varphi(x-0) = 1$ at x, which gives the first formula. For $x=a$ or $x=b$ the function φ is differentiable on (a,b) and its derivative is equal to $\frac{-1}{b-a}$. It has a jump of $\varphi(a+0)-\varphi(a)=1$ at the point a, while $\varphi(b)-\varphi(b-0)=0$, which gives the second formula. □

Here, as in the rest of the book, we write $\int_0^1 g(t)d\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi:[0,1]\to\mathbb{R}$ of bounded variation, and $\int_0^1 g(t)dt$ for the Riemann integral.

Theorem 1.1 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[a,b]$ for some $n\geq 1$. Then for every $x\in[a,b]$ we have*

$$\frac{1}{b-a}\int_a^b f(t)dt \;=\; f(x)-T_n(x)+R_n^1(x), \tag{1.1}$$

$$\frac{1}{b-a}\int_a^b f(t)dt \;=\; f(x)-T_{n-1}(x)+R_n^2(x) \tag{1.2}$$

where $T_0(x)=0$, and for $1\leq m\leq n$

$$T_m(x)=\sum_{k=1}^{m}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right], \tag{1.3}$$

$$R_n^1(x)=\frac{(b-a)^{n-1}}{n!}\int_a^b B_n^*\left(\frac{x-t}{b-a}\right)df^{(n-1)}(t),$$

$$R_n^2(x)=\frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right]df^{(n-1)}(t).$$

Proof. Using integration by parts we have

$$\begin{aligned}R_k^1(x) \;&=\; \frac{(b-a)^{k-1}}{k!}B_k^*\left(\frac{x-t}{b-a}\right)f^{(k-1)}(t)\Big|_a^b \\ &-\; \frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t)dB_k^*\left(\frac{x-t}{b-a}\right).\end{aligned} \tag{1.4}$$

For every $k\geq 1$ and every $x\in[a,b)$ we have

$$B_k^*\left(\frac{x-b}{b-a}\right)=B_k^*\left(\frac{x-a}{b-a}-1\right)=B_k^*\left(\frac{x-a}{b-a}\right)=B_k\left(\frac{x-t}{b-a}\right). \tag{1.5}$$

Also, for $k\geq 2$ the above formula is valid for every $x\in[a,b]$. The identity (1.4) for $k=1$ becomes

$$R_1^1(x)=B_1^*\left(\frac{x-t}{b-a}\right)f(t)\Big|_a^b-\int_{[a,b]}f(t)dB_1^*\left(\frac{x-t}{b-a}\right).$$

If $x\in[a,b)$, then using Lemma 1.1 and (1.5) we get

$$\begin{aligned}R_1^1(x) \;&=\; B_1\left(\frac{x-a}{b-a}\right)[f(b)-f(a)]+\frac{1}{b-a}\int_a^b f(t)dt-f(x)\\ &=\; T_1(x)+\frac{1}{b-a}\int_a^b f(t)dt-f(x).\end{aligned}$$

If $x = b$, then using Lemma 1.1 we get

$$\begin{aligned}
R_1^1(b) &= B_1^*(0)f(b) - B_1^*(1)f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\
&= -\frac{1}{2}f(b) + \frac{1}{2}f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\
&= \frac{1}{2}[f(b) - f(a)] + \frac{1}{b-a}\int_a^b f(t)dt - f(b) \\
&= T_1(b) + \frac{1}{b-a}\int_a^b f(t)dt - f(b).
\end{aligned}$$

So, for every $x \in [a,b]$ we have

$$R_1^1(x) = T_1(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x), \tag{1.6}$$

which is just the identity (1.1) for $n = 1$. Further, for every $k \geq 2$

$$\frac{d}{dt}B_k^*\left(\frac{x-t}{b-a}\right) = -\frac{k}{b-a}B_{k-1}^*\left(\frac{x-t}{b-a}\right),$$

except for t from discrete set $x + (b-a)\mathbb{Z} \subset \mathbb{R}$, since the Bernoulli polynomials satisfy $\frac{d}{dt}B_k(t) = kB_{k-1}(t)$. Using the above formula and the fact that $B_k^*\left(\frac{x-t}{b-a}\right)$ is continuous for $k \geq 2$, we get

$$\begin{aligned}
&-\frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t)dB_k^*\left(\frac{x-t}{b-a}\right) \\
&\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_a^b B_{k-1}^*\left(\frac{x-t}{b-a}\right) f^{(k-1)}(t)dt \\
&\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_{[a,b]} B_{k-1}^*\left(\frac{x-t}{b-a}\right) df^{(k-2)}(t) \\
&\quad = R_{k-1}^1(x).
\end{aligned}$$

Using this formula and (1.5), from (1.4) we get the identity

$$R_k^1(x) = \frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] + R_{k-1}^1(x),$$

which holds for $k = 2,\ldots,n$ and for every $x \in [a,b]$. So, for $n \geq 2$ and for every $x \in [a,b]$ we get

$$R_n^1(x) = \sum_{k=2}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)[f^{(k-1)}(b) - f^{(k-1)}(a)] + R_1^1(x),$$

which, in combination with (1.6), yields (1.1).

To obtain the identity (1.2), note that

$$
\begin{aligned}
R_n^2(x) &= R_n^1(x) - \frac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-a}{b-a}\right) \int_{[a,b]} d f^{(n-1)}(t) \\
&= R_n^1(x) - \frac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-a}{b-a}\right) [f^{(n-1)}(b) - f^{(n-1)}(a)] \\
&= R_n^1(x) + T_n(x) - T_{n-1}(x),
\end{aligned}
$$

and apply (1.1). □

1.2 General Euler-Ostrowski formulae

The main results of this section are the general Euler-Ostrowski formulae which generalize extended Euler identities (1.1) and (1.2), in a sense that the value of the integral is approximated by the values of the function in m equidistant points, instead of by its value in just one point. The results presented in this section were published in [63].

To derive these formulae, we will need an analogue of Multiplication Theorem, stated for periodic functions B_n^*. Multiplication Theorem for Bernoulli polynomials B_n states (cf. [1] or Appendix):

$$
B_n(mt) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1 \tag{1.7}
$$

That (1.7) is true for $B_n^*(t)$ and $t \in [0, 1/m)$ is obvious. For $t \in [j/m, (j+1)/m)$, $1 \leq j \leq m-1$:

$$
\begin{aligned}
B_n^*(mt) &= B_n^*(m(t - j/m)) = m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k-j}{m}\right) \\
&= m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k}{m}\right),
\end{aligned}
$$

so the statement is true again. Thus, we have

$$
B_n^*(mt) = m^{n-1} \sum_{k=0}^{m-1} B_n^*\left(t + \frac{k}{m}\right), \quad n \geq 0, \ m \geq 1. \tag{1.8}
$$

Interval $[0, 1]$ is used for simplicity and involves no loss in generality.

The following theorem is crucial for our further investigations but is also of independent interest. Namely, the remainder is expressed in terms of $B_n^*(x - mt)$.

Theorem 1.2 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous of bounded variation on $[0,1]$ for some $n\geq 1$. Then, for $x\in[0,1]$ and $m\in\mathbb{N}$, we have*

$$\int_0^1 f(t)dt = \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_n(x) + \frac{1}{n!\cdot m^n}\int_0^1 B_n^*(x-mt)\,df^{(n-1)}(t), \tag{1.9}$$

where

$$T_n(x) = \sum_{j=1}^{n} \frac{B_j(x)}{j!\cdot m^j}[f^{(j-1)}(1)-f^{(j-1)}(0)]$$

Proof. From (1.8) we get

$$B_n^*(x-mt) = m^{n-1}\sum_{k=0}^{m-1} B_n^*\left(\frac{x+k}{m}-t\right)$$

Multiplying this with $df^{(n-1)}(t)$ and integrating over $[0,1]$ produces formula (1.9) after applying (1.1). □

Formula (1.9) can easily be rewritten as:

$$\begin{aligned}\int_0^1 f(t)dt &= \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) - T_{n-1}(x)\\ &+ \frac{1}{n!\cdot m^n}\int_0^1 [B_n^*(x-mt)-B_n(x)]\,df^{(n-1)}(t),\end{aligned} \tag{1.10}$$

with $T_0(x)=0$.

We call formulae (1.9) and (1.10) the general Euler-Ostrowski formulae.

Theorem 1.3 *Assume (p,q) is a pair of conjugate exponents, that is $1\leq p,\ q\leq\infty$, $1/p+1/q=1$. Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)}\in L_p[0,1]$ for some $n\geq 1$. Then, for $x\in[0,1]$ and $m\in\mathbb{N}$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x)\right| \leq K(n,q)\cdot\|f^{(n)}\|_p, \tag{1.11}$$

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x)\right| \leq K^*(n,q)\cdot\|f^{(n)}\|_p, \tag{1.12}$$

where

$$K(n,q) = \frac{1}{n!\cdot m^n}\left[\int_0^1 |B_n^*(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}},$$

$$K^*(n,q) = \frac{1}{n!\cdot m^n}\left[\int_0^1 |B_n^*(t)-B_n(x)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}.$$

These inequalities are sharp for $1<p\leq\infty$ and the best possible for $p=1$.

Proof. Inequalities (1.11) and (1.12) follow immediately after applying Hölder's inequality to the remainders in formulae (1.9) and (1.10) and using the fact that functions $B_n^*(t)$ are periodic. To prove that the inequalities are sharp, put

$$\begin{aligned}
&f^{(n)}(t) = \operatorname{sgn} B_n^*(x-mt) \cdot |B_n^*(x-mt)|^{1/(p-1)} \quad \text{for} \quad 1<p<\infty \quad \text{and} \\
&f^{(n)}(t) = \operatorname{sgn} B_n^*(x-mt) \quad \text{for} \quad p=\infty \quad \text{in} \quad (1.11), \\
&f^{(n)}(t) = \operatorname{sgn}(B_n^*(x-mt) - B_n(x)) \cdot |B_n^*(x-mt) - B_n(x)|^{1/(p-1)} \\
&\text{for} \ \ 1<p<\infty \quad \text{and} \\
&f^{(n)}(t) = \operatorname{sgn}(B_n^*(x-mt) - B_n(x)) \quad \text{for} \quad p=\infty \quad \text{in} \quad (1.12).
\end{aligned}$$

For $p=1$ it is easy to see that

$$\left| \int_0^1 B_n^*(x-mt) f^{(n)}(t)dt \right| \le \max_{t\in[0,1]} |B_n^*(t)| \int_0^1 \left| f^{(n)}(t) \right| dt$$

is the best possible inequality (compare with the proof of Theorem 2.2 in Section 2.3). □

Corollary 1.1 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n)} \in L_\infty[0,1]$. Let $x\in[0,1]$. If n is odd, then we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \le \frac{(4-2^{1-n})|B_{n+1}|}{m^n\cdot(n+1)!} \cdot \|f^{(n)}\|_\infty, \tag{1.13}$$

and for $n=1$

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \right| \le \frac{1}{m}\left[\frac{1}{4} + \left(x-\frac{1}{2}\right)^2\right] \cdot \|f'\|_\infty, \tag{1.14}$$

while for $n\ge 3$

$$\begin{aligned}
&\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \qquad (1.15)\\
&\le \frac{\|f^{(n)}\|_\infty}{m^n\cdot n!}\left((1-2|x-x_1|)\cdot|B_n(x)| + \frac{2}{n+1}|B_{n+1}(x) - B_{n+1}(x_1)| \right),
\end{aligned}$$

where $x_1\in[0,1]$ is such that $B_n(x_1)=B_n(x)$ and $x_1\ne x$, except when $B_{n-1}(x)=0$. If $x=0$ or $x=1$, take $x_1=1/2$.

If n is even, then we have

$$\begin{aligned}
&\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| \qquad (1.16)\\
&\le \frac{4\|f^{(n)}\|_\infty}{m^n\cdot(n+1)!}\cdot|B_{n+1}(x_1)| = \frac{4\|f^{(n)}\|_\infty}{m^n\cdot(n+1)!}\max_{t\in[0,1]}|B_{n+1}(t)|,
\end{aligned}$$

where $B_n(x_1) = 0$, *and*

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x) \right| \tag{1.17}$$
$$\leq \frac{\|f^{(n)}\|_\infty}{m^n \cdot n!}\left((-1)^{n/2}\left(1-4\left|x-1/2\right|\right)B_n(x) + \frac{4}{n+1}\left|B_{n+1}(x)\right|\right).$$

Proof. Put $p = \infty$ in Theorem 1.3. Inequality (1.13) follows straightforward since it is known that, for an odd n, Bernoulli polynomials have constant sign on $(0,1/2)$ and on $(1/2,1)$. (1.14) also follows by direct calculation.

To prove (1.15), assume first that $0 \leq x \leq 1/2$. For an odd n we have $B_n(1-t) = -B_n(t)$, so we can rewrite $K^*(n,1)$ as

$$\int_0^{1/2} \left|B_n(t) - B_n(x)\right| dt + \int_0^{1/2} \left|B_n(t) + B_n(x)\right| dt.$$

The second integral has no zeros on $(0,1/2)$, so we can calculate it easily. The first integral, however, has two zeros. One is obviously x and the other is x_1, where $x_1 \in [0,1/2]$ and $B_n(x_1) = B_n(x)$. When $1/2 \leq x \leq 1$, the statement follows similarly.

Next, assume $0 \leq x \leq 1/2$. Since $B_n(t)$ are symmetric about $t = 1/2$ for an even n, we can rewrite $K^*(n,1)$ as $2\int_0^{1/2} \left|B_n(t) - B_n(x)\right| dt$. As Bernoulli polynomials are monotonous on $(0,1/2)$ for an even n, inequality (1.17) follows. For $1/2 \leq x \leq 1$ the statement follows analogously. Using similar arguments we get (1.16). □

Remark 1.1 For $m = 1$, formulae (1.9) and (1.10) reduce to (1.1) and (1.2), and thus give all the results from [30] i.e. the generalizations of Ostrowski's inequality; especially, (1.14) produces the classical Ostrowski's inequality for $m = 1$.

For $m = 1$ and $n = 2$, (1.17) gives an improvement of a result obtained in [38]. This was discussed in detail in [30].

Further, taking $m = 1$ and $n = 3$ in (1.15) produces a result obtained in [4]. These results are therefore a generalization of the results from that paper.

Corollary 1.2 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_1[0,1]$ *and* $x \in [0,1]$. *For* $n = 1$, *we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + \frac{B_1(x)}{m}[f(1) - f(0)] \right| \leq \frac{\|f'\|_1}{2m}, \tag{1.18}$$

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) \right| \leq \frac{\|f'\|_1}{m}\left(\frac{1}{2} + \left|x - \frac{1}{2}\right|\right), \tag{1.19}$$

For an odd n, $n \geq 3$, *we have*

$$\left| \int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x) \right| < \frac{2\|f^{(n)}\|_1}{(1-2^{-n-1})(2\pi m)^n}, \tag{1.20}$$

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x)\right| \tag{1.21}$$
$$< \frac{\|f^{(n)}\|_1}{m^n \cdot n!}\left(\frac{2n!}{(1-2^{-n-1})(2\pi)^n} + |B_n(x)|\right),$$

If n is even, then we have

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x)\right| \le \frac{|B_n|}{m^n \cdot n!} \cdot \|f^{(n)}\|_1, \tag{1.22}$$

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x)\right| \tag{1.23}$$
$$\le \frac{\|f^{(n)}\|_1}{m^n \cdot n!}\left((1-2^{-n})|B_n| + \left|2^{-n}B_n - B_n(x)\right|\right).$$

Proof. Put $p = 1$ in Theorem 1.3. Inequalities (1.18) and (1.19) follow by direct calculation. Using estimations of the maximal value of Bernoulli polynomials (cf. [1]), we get (1.20), (1.21) and (1.22). Finally, since $B_n(t)$ are symmetric about $t = 1/2$ for an even n, it is enough to consider them on $(0,1/2)$ and there they are monotonous. So the maximal value of $|B_n(t) - B_n(x)|$ is obtained either for $t = 0$ or for $t = 1/2$. Using formula

$$\max\{|A|,|B|\} = \frac{1}{2}\left(|A+B| + |A-B|\right),$$

(1.23) follows. □

Corollary 1.3 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* $f^{(n)} \in L_2[0,1]$ *and* $x \in [0,1]$. *Then we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_n(x)\right| \le \frac{\|f^{(n)}\|_2}{m^n}\left(\frac{|B_{2n}|}{(2n)!}\right)^{1/2}, \tag{1.24}$$

$$\left|\int_0^1 f(t)dt - \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right) + T_{n-1}(x)\right| \tag{1.25}$$
$$\le \frac{\|f^{(n)}\|_2}{m^n \cdot n!}\left(\frac{(n!)^2}{(2n)!}|B_{2n}| + B_n^2(x)\right)^{1/2},$$

Proof. Both inequalities follow by direct calculation after taking $p = 2$ in Theorem 1.3. □

It is interesting to consider which $x \in [0,1]$ gives the optimal estimation in inequalities (1.15) and (1.17). In (1.14) it is obvious that $x = 1/2$ is such point. Differentiating the function on the right-hand side of (1.17) – this is the case when n is even – it is easy to see that it obtains its minimum for $x = 1/4$ and $x = 3/4$ (for $n \ge 2$) while its maximal value is in $x = 0$ and $x = 1$ (for $n \geq 4$). Of course, the minimal value is of greater interest. In that case, the quadrature formulae take the following form

$$\int_0^1 f(t)dt \approx \frac{1}{4}\left(f(1) + 4f\left(\frac{1}{4}\right) - f(0)\right)$$
$$\int_0^1 f(t)dt \approx \frac{1}{4}\left(f(0) + 4f\left(\frac{3}{4}\right) - f(1)\right)$$

Also, if we take these parameters and put them in (1.10), then add them up and divide by 2, we get a two-point formula where the integral is approximated by values of the function in $x = 1/4$ and $x = 3/4$. The error estimation for this formula can be deduced from the following, more general, estimation. Using triangle inequality, we get

$$\begin{aligned}
&\left|\int_0^1 f(t)dt - \frac{1}{2m}\sum_{k=0}^{m-1}\left(f\left(\frac{x+k}{m}\right) + f\left(\frac{1-x+k}{m}\right)\right)\right. \\
&\left. + \sum_{j=1}^{(n-2)/2} \frac{B_{2j}(x)}{(2j)! \cdot m^{2j}}[f^{(2j-1)}(1) - f^{(2j-1)}(0)]\right| \\
&\leq \frac{\|f^{(n)}\|_\infty}{m^n \cdot n!}\left((-1)^{n/2}(1 - 4|x - 1/2|)B_n(x) + \frac{4}{n+1}|B_{n+1}(x)|\right).
\end{aligned}$$

Therefore, this formula gives the best error estimate for $x = 1/4$.

On the other hand, inequality (1.15) behaves quite oppositely (this is the case when n is odd and $n \geq 3$). Observe that x_1 is a decreasing function of x and it is differentiable on $(0, 1/2)$. This is sufficient since the function on the right-hand side of that inequality (denote it by $F(x)$) obtains the same value for x and $1 - x$. For $0 \leq x \leq 1/2$, we get

$$F'(x) = (-1)^{(n+1)/2} \cdot n(1 - 2|x - x_1|)B_{n-1}(x).$$

Since $F'(x)$ changes sign from positive to negative when passing through point $\alpha \in (0, 1/2)$ such that $B_{n-1}(\alpha) = 0$, we conclude that $F(x)$ obtains maximal value at α. Note that α is close to $1/4$, but a bit smaller. Minimum is then obtained at the end points of the interval i.e. for $x = 0$ and $x = 1/2$ (the same value is obtained at both of these points).

1.2.1 Trapezoid formula

Choosing $x = 0$ and $x = 1$ in (1.9) and (1.10) when $m = 1$, adding those two formulae up and then dividing the resulting formula by 2, produces the Euler trapezoid formulae - and all the other results - obtained in [25]. Here, we just state the error estimates for this type of quadrature formulae. Namely, for $p = \infty$ and $p = 1$, we have:

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0) + f(1)]\right| \leq C_T(m, q) \cdot \|f^{(m)}\|_p, \quad m = 1, 2$$

where

$$C_T(1,1) = \frac{1}{4}, \quad C_T(1,\infty) = \frac{1}{2}, \quad C_T(2,1) = \frac{1}{12}, \quad C_T(2,\infty) = \frac{1}{8},$$

while for $m = 2,3,4$

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + \frac{1}{12}[f'(1)-f'(0)]\right| \leq C_T(m,q)\cdot \|f^{(m)}\|_p,$$

where

$$C_T(2,1) = \frac{1}{18\sqrt{3}}, \quad C_T(3,1) = \frac{1}{192}, \quad C_T(4,1) = \frac{1}{720},$$
$$C_T(2,\infty) = \frac{1}{12}, \quad C_T(3,\infty) = \frac{1}{72\sqrt{3}}, \quad C_T(4,\infty) = \frac{1}{384}.$$

1.2.2 Midpoint formula

For $m = 1$ and $x = 1/2$ in (1.9) and (1.10), we get the Euler midpoint formulae derived in [23] and of course all other results from that paper follow directly. The error estimates for this type of quadrature formulae, for $p = \infty$ and $p = 1$, are:

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq C_M(m,q)\cdot \|f^{(m)}\|_p, \quad m = 1,2$$

where

$$C_M(1,1) = \frac{1}{4}, \quad C_M(1,\infty) = \frac{1}{2}, \quad C_M(2,1) = \frac{1}{24}, \quad C_M(2,\infty) = \frac{1}{8},$$

while for $m = 2,3,4$

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) - \frac{1}{24}[f'(1)-f'(0)]\right| \leq C_M(m,q)\cdot \|f^{(m)}\|_p,$$

where

$$C_M(2,1) = \frac{1}{18\sqrt{3}}, \quad C_M(3,1) = \frac{1}{192}, \quad C_M(4,1) = \frac{7}{5760},$$
$$C_M(2,\infty) = \frac{1}{12}, \quad C_M(3,\infty) = \frac{1}{72\sqrt{3}}, \quad C_M(4,\infty) = \frac{1}{384}.$$

Chapter 2

Euler two-point formulae

2.1 Introduction

In this chapter we study, for each real number $x \in [0,1/2]$, the general two-point quadrature formula

$$\int_0^1 f(t)dt = \frac{1}{2}[f(x)+f(1-x)]+E(f;,x) \tag{2.1}$$

with $E(f;x)$ being the remainder. This family of two-point quadrature formulae was considered by Guessab and Schmeisser in [68] and they established sharp estimates for the remainder under various regularity conditions. The aim of this chapter is to establish general two-point formula (2.1) using identities (1.1) and (1.2) and give various error estimates for the quadrature rules based on such generalizations. In Section 2 we use the extended Euler formulae to obtain two new integral identities. We call them the general Euler two-point formulae. In Section 3, we prove a number of inequalities which give error estimates for the general Euler two-point formulae for functions whose derivatives are from the L_p-spaces, thus we extend the results from [68] and we generalize the results from papers [25]-[27], [83] and [84]. These inequalities are generally sharp (in case $p=1$ the best possible). Special attention is devoted to the case where we have some boundary conditions and in some cases we compare our estimates with the Fink's estimates ([68], [45]). In Section 4 we give a variant of the inequality proved in the paper [91] and we use those results to prove some inequalities for the general Euler two-point formula. The general Euler two-point formulae are used in Section 5 with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration. In Section 6 we generalize estimation of two-point formula by using pre-Grüss inequality and in Section 7 we give Hermite-Hadamard's inequalities of Bullen type.

2.2 General Euler two-point formulae

The results from this and next section are published in [98].

For $k \geq 1$ and fixed $x \in [0,1/2]$ define the functions $G_k^x(t)$ and $F_k^x(t)$ as

$$G_k^x(t) = B_k^*(x-t) + B_k^*(1-x-t), \; t \in \mathbb{R}$$

and $F_k^x(t) = G_k^x(t) - \tilde{B}_k(x), \; t \in \mathbb{R}$, where

$$\tilde{B}_k(x) = B_k(x) + B_k(1-x), \; x \in [0,1/2], \; k \geq 1.$$

Especially, we get $\tilde{B}_1(x) = 0, \; \tilde{B}_2(x) = 2x^2 - 2x + 1/3, \; \tilde{B}_3(x) = 0$. Also, for $k \geq 2$ we have $\tilde{B}_k(x) = G_k^x(0)$, that is $F_k^x(t) = G_k^x(t) - G_k^x(0), \; k \geq 2$, and $F_1^x(t) = G_1^x(t), \; t \in \mathbb{R}$. Obviously, $G_k^x(t)$ and $F_k^x(t)$ are periodic functions of period 1 and continuous for $k \geq 2$.

Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geq 1$. We introduce the following notation for each $x \in [0,1/2]$

$$D(x) = \frac{1}{2}\left[f(x) + f(1-x)\right].$$

Further, we define $\tilde{T}_0(x) = 0$ and, for $1 \leq m \leq n, \; x \in [0,1/2]$

$$\tilde{T}_m(x) = \frac{1}{2}\left[T_m(x) + T_m(1-x)\right],$$

where $T_m(x)$ is given by (1.3). It is easy to see that

$$\tilde{T}_m(x) = \frac{1}{2}\sum_{k=1}^{m} \frac{\tilde{B}_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{2.2}$$

In the next theorem we establish two formulae which play the key role in this chapter. We call them the general Euler two-point formulae.

Theorem 2.1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Then for each $x \in [0,1/2]$*

$$\int_0^1 f(t)\mathrm{d}t = D(x) - \tilde{T}_n(x) + \tilde{R}_n^1(f) \tag{2.3}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(x) - \tilde{T}_{n-1}(x) + \tilde{R}_n^2(f), \tag{2.4}$$

where

$$\tilde{R}_n^1(f) = \frac{1}{2(n!)}\int_0^1 G_n^x(t)\,\mathrm{d}f^{(n-1)}(t), \;\; \tilde{R}_n^2(f) = \frac{1}{2(n!)}\int_0^1 F_n^x(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put $x \equiv x$ and $x \equiv 1-x$ in formula (1.1) to get two new formulae. Then multiply these new formulae by $1/2$ and add them up. The result is formula (2.3). Formula (2.4) is obtained from (1.2) by the same procedure. □

Remark 2.1 If in Theorem 2.1 we choose $x = 0, 1/2, 1/3, 1/4$ we get Euler trapezoid [25], Euler midpoint [23], Euler two-point Newton-Cotes [84] and Euler two-point Maclaurin formulae respectively.

By direct calculations for each $x \in [0, 1/2]$, we get

$$F_1^x(t) = G_1^x(t) = \begin{cases} -2t, & 0 \le t \le x \\ -2t+1, & x < t \le 1-x \\ -2t+2, & 1-x < t \le 1 \end{cases}, \tag{2.5}$$

$$G_2^x(t) = \begin{cases} 2t^2+2x^2-2x+1/3, & 0 \le t \le x \\ 2t^2-2t+2x^2+1/3, & x < t \le 1-x \\ 2t^2-4t+2x^2-2x+7/3, & 1-x < t \le 1 \end{cases}, \tag{2.6}$$

$$F_2^x(t) = \begin{cases} 2t^2, & 0 \le t \le x \\ 2t^2-2t+2x, & x < t \le 1-x \\ 2t^2-4t+2, & 1-x < t \le 1 \end{cases} \tag{2.7}$$

and

$$F_3^x(t) = G_3^x(t) = \begin{cases} -2t^3+(-6x^2+6x-1)t, & 0 \le t \le x \\ -2t^3+3t^2+(-6x^2-1)t+3x^2, & x < t \le 1-x \\ -2t^3+6t^2+(-6x^2+6x-7)t & \\ \qquad +6x^2-6x+3, & 1-x < t \le 1. \end{cases} \tag{2.8}$$

Now, we will prove some properties of the functions $G_k^x(t)$ and $F_k^x(t)$ defined above. The Bernoulli polynomials are symmetric with respect to $1/2$, that is

$$B_k(1-x) = (-1)^k B_k(x), \quad k \ge 1. \tag{2.9}$$

Also, we have $B_k(1) = B_k(0) = B_k$, $k \ge 2$, $B_1(1) = -B_1(0) = 1/2$ and $B_{2j-1} = 0$, $j \ge 2$. Therefore, we get $\tilde{B}_{2j-1}(x) = 0$, $j \ge 1$ and $\tilde{B}_{2j}(x) = 2B_{2j}(x)$, $x \in [0, 1/2]$. Now, we have $F_{2j-1}^x(t) = G_{2j-1}^x(t)$, $j \ge 1$, and

$$F_{2j}^x(t) = G_{2j}^x(t) - \tilde{B}_{2j}(x) = G_{2j}^x(t) - 2B_{2j}(x), \; x \in [0, 1/2], \quad j \ge 1. \tag{2.10}$$

Further, the points 0 and 1 are zeros of $F_k^x(t) = G_k^x(t) - G_k^x(0)$, $k \ge 2$, that is $F_k^x(0) = F_k^x(1) = 0$, $k \ge 1$. As we shall see below, 0 and 1 are the only zeros of $F_{2j}^x(t)$ for $j \ge 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$. Next, setting $t = 1/2$ in (2.9) we get $B_k(1/2) = (-1)^k B_k(1/2)$, $k \ge 1$, which implies that $B_{2j-1}(1/2) = 0$, $j \ge 1$. Using the above formulae, we get $F_{2j-1}^x(1/2) = G_{2j-1}^x(1/2) = 0$, $j \ge 1$. We shall see that 0, 1/2 and 1 are the only zeros of $F_{2j-1}^x(t) = G_{2j-1}^x(t)$, for $j \ge 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$. Also, note that for $x \in [0, 1/2]$, $j \ge 1$ $G_{2j}^x(1/2) = 2B_{2j}(1/2 - x)$ and

$$F_{2j}^x(1/2) = G_{2j}^x(1/2) - \tilde{B}_{2j}(x) = 2B_{2j}(1/2 - x) - 2B_{2j}(x). \tag{2.11}$$

Lemma 2.1 *For $k \geq 2$ we have $G_k^x(1-t) = (-1)^k G_k^x(t)$, $0 \leq t \leq 1$ and $F_k^x(1-t) = (-1)^k F_k^x(t)$, $0 \leq t \leq 1$.*

Proof. As the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geq 2$. Therefore, for $k \geq 2$ and $0 \leq t \leq 1$ we have

$$
\begin{aligned}
G_k^x(1-t) &= B_k^*(x-1+t) + B_k^*(-x+t) \\
&= \begin{cases} B_k(x+t) + B_k(1-x+t), & 0 \leq t \leq x, \\ B_k(x+t) + B_k(-x+t), & x < t \leq 1-x, \\ B_k(-1+x+t) + B_k(-x+t), & 1-x < t \leq 1, \end{cases} \\
&= (-1)^k \times \\
&\quad \begin{cases} B_k(1-x-t) + B_k(x-t), & 0 \leq t \leq x, \\ B_k(1-x-t) + B_k(1+x-t), & x < t \leq 1-x, \\ B_k(2-x-t) + B_k(1+x-t), & 1-x < t \leq 1, \end{cases} \\
&= (-1)^k G_k^x(t),
\end{aligned}
$$

which proves the first identity. Further, we have $F_k^x(t) = G_k^x(t) - G_k^x(0)$ and $(-1)^k G_k^x(0) = G_k^x(0)$, since $G_{2j+1}^x(0) = 0$, so that we have

$$F_k^x(1-t) = G_k^x(1-t) - G_k^x(0) = (-1)^k [G_k^x(t) - G_k^x(0)] = (-1)^k F_k^x(t),$$

which proves the second identity. □

Note that the identities established in Lemma 2.1 are valid for $k = 1$, too, except at the points x, and $1-x$ of discontinuity of $F_1^x(t) = G_1^x(t)$.

Lemma 2.2 *For $k \geq 2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ the function $G_{2k-1}^x(t)$ has no zeros in the interval $(0, 1/2)$. For $0 < t < 1/2$ the sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}^x(t) > 0,\ x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \text{ and } (-1)^k G_{2k-1}^x(t) > 0,\ x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right].$$

Proof. For $k = 2$, $G_3^x(t)$ is given by (2.8) and it is easy to see that for each $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$

$$G_3^x(t) < 0,\ 0 < t < \frac{1}{2}$$

and for each $x \in \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$

$$G_3^x(t) > 0,\ 0 < t < \frac{1}{2}.$$

Thus, our assertion is true for $k = 2$. Now, assume that $k \geq 3$. Then $2k-1 \geq 5$ and $G_{2k-1}^x(t)$ is continuous and at least twice differentiable function. Using (A-2) we get

$$G^{x\prime}_{2k-1}(t) = -(2k-1) G_{2k-2}^x(t)$$

and

$$G^{x\prime\prime}_{2k-1}(t) = (2k-1)(2k-2)G^x_{2k-3}(t).$$

Let us suppose that G^x_{2k-3} has no zeros in the interval $\left(0,\frac{1}{2}\right)$. We know that 0 and $\frac{1}{2}$ are zeros of $G^x_{2k-1}(t)$. Let us suppose that some α, $0<\alpha<\frac{1}{2}$, is also a zero of $G^x_{2k-1}(t)$. Then inside each of the intervals $(0,\alpha)$ and $\left(\alpha,\frac{1}{2}\right)$ the derivative $G^{x\prime}_{2k-1}(t)$ must have at least one zero, say β_1, $0<\beta_1<\alpha$ and β_2, $\alpha<\beta_2<\frac{1}{2}$. Therefore, the second derivative $G^{x\prime\prime}_{2k-1}(t)$ must have at least one zero inside the interval (β_1,β_2). Thus, from the assumption that $G^x_{2k-1}(t)$ has a zero inside the interval $\left(0,\frac{1}{2}\right)$, it follows that $(2k-1)(2k-2)G^x_{2k-3}(t)$ also has a zero inside this interval. Thus, $G^x_{2k-1}(t)$ can not have a zero inside the interval $\left(0,\frac{1}{2}\right)$. To determine the sign of $G^x_{2k-1}(t)$, note that

$$G^x_{2k-1}(x) = B_{2k-1}(1-2x).$$

We have [1, 23.1.14]

$$(-1)^k B_{2k-1}(t) > 0,\ 0<t<\frac{1}{2},$$

which implies for $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$

$$(-1)^{k-1}G^x_{2k-1}(x) = (-1)^{k-1}B_{2k-1}(1-2x) = (-1)^k B_{2k-1}(2x) > 0$$

and

$$(-1)^k G^x_{2k-1}(x) = (-1)^k B_{2k-1}(1-2x) > 0 \text{ for } x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right].$$

Consequently, we have

$$(-1)^{k-1}G^x_{2k-1}(t) > 0,\ 0<t<\frac{1}{2} \text{ for } x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$$

and

$$(-1)^k G^x_{2k-1}(t) > 0,\ 0<t<\frac{1}{2} \text{ for } x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right].$$

□

Corollary 2.1 *For $k\geq 2$ and $x\in\left[0,\frac{1}{2}-\frac{1}{2\sqrt{3}}\right]$ the functions $(-1)^k F^x_{2k}(t)$ and $(-1)^k G^x_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Also, for $x\in\left[\frac{1}{2\sqrt{3}},\frac{1}{2}\right]$ the functions $(-1)^{k-1}F^x_{2k}(t)$ and $(-1)^{k-1}G^x_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$, and strictly decreasing on the interval $(1/2,1)$. Further, for $k\geq 2$, we have*

$$\max_{t\in[0,1]}\left|F^x_{2k}(t)\right| = 2\left|B_{2k}(1/2-x) - B_{2k}(x)\right|$$

and

$$\max_{t\in[0,1]}\left|G^x_{2k}(t)\right| = 2\max\left\{\left|B_{2k}(x)\right|,\left|B_{2k}(1/2-x)\right|\right\}.$$

Proof. Using (A-2) we get

$$\left[(-1)^k F^x_{2k}(t)\right]' = \left[(-1)^k G^x_{2k}(t)\right]' = 2k(-1)^{k-1}G^x_{2k-1}(t)$$

and $(-1)^{k-1}G^x_{2k-1}(t) > 0$ for $0 < t < 1/2$ and $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right]$, by Lemma 2.2. Thus, $(-1)^k F^x_{2k}(t)$ and $(-1)^k G^x_{2k}(t)$ are strictly increasing on the interval $(0,1/2)$. Also, by Lemma 2.1, we have $F^x_{2k}(1-t) = F^x_{2k}(t)$, $0 \le t \le 1$ and $G^x_{2k}(1-t) = G^x_{2k}(t)$, $0 \le t \le 1$, which implies that $(-1)^k F^x_{2k}(t)$ and $(-1)^k G^x_{2k}(t)$ are strictly decreasing on the interval $(1/2,1)$. The proof of second statement is similar. Further, $F^x_{2k}(0) = F^x_{2k}(1) = 0$, which implies that $\left|F^x_{2k}(t)\right|$ achieves its maximum at $t = 1/2$, that is

$$\max_{t\in[0,1]} |F^x_{2k}(t)| = |F^x_{2k}(1/2)| = 2\,|B_{2k}(1/2-x) - B_{2k}(x)|\,.$$

Also

$$\max_{t\in[0,1]} |G^x_{2k}(t)| = \max\{|G^x_{2k}(0)|, |G^x_{2k}(1/2)|\} = 2\max\{|B_{2k}(x)|, |B_{2k}(1/2-x)|\}, \quad (2.12)$$

which completes the proof. □

Corollary 2.2 *For* $k \ge 2$, *and* $x \in \left[0, \frac{1}{2} - \frac{1}{2\sqrt{3}}\right] \cup \left[\frac{1}{2\sqrt{3}}, \frac{1}{2}\right]$ *we have*

$$\int_0^1 \left|F^x_{2k-1}(t)\right| \mathrm{d}t = \int_0^1 \left|G^x_{2k-1}(t)\right| \mathrm{d}t = \frac{2}{k}\,|B_{2k}(1/2-x) - B_{2k}(x)|\,.$$

Also, we have

$$\int_0^1 |F^x_{2k}(t)|\,\mathrm{d}t = \left|\tilde{B}_{2k}(x)\right| = 2\,|B_{2k}(x)| \text{ and } \int_0^1 |G^x_{2k}(t)|\,\mathrm{d}t \le 2\left|\tilde{B}_{2k}(x)\right| = 4\,|B_{2k}(x)|\,.$$

Proof. Using Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned}\int_0^1 \left|G^x_{2k-1}(t)\right| \mathrm{d}t &= 2\left|\int_0^{1/2} G^x_{2k-1}(t)\mathrm{d}t\right| = 2\left|-\frac{1}{2k}\,G^x_{2k}(t)\Big|_0^{1/2}\right| \\ &= \frac{1}{k}\,|G^x_{2k}(1/2) - G^x_{2k}(0)| = \frac{2}{k}\,|B_{2k}(1/2-x) - B_{2k}(x)|,\end{aligned}$$

which proves the first assertion. By Corollary 2.1 and because $F^x_{2k}(0) = F^x_{2k}(1) = 0$, $F^x_{2k}(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (2.10) we get

$$\begin{aligned}\int_0^1 |F^x_{2k}(t)|\,\mathrm{d}t &= \left|\int_0^1 F^x_{2k}(t)\mathrm{d}t\right| = \left|\int_0^1 \left[G^x_{2k}(t) - \tilde{B}_{2k}(x)\right]\mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\,G^x_{2k+1}(t)\Big|_0^1 - \tilde{B}_{2k}(x)\right| = \left|\tilde{B}_{2k}(x)\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (2.10) again and the triangle inequality to obtain the third formula. □