Chapter 1

Hilbert-type inequalities with conjugate exponents

One of the most important inequalities in modern mathematics is the well-known Hilbert inequality. Applications of this inequality in various branches of mathematics have certainly contributed to its importance. David Hilbert was the first mathematician who started to deal with the Hilbert inequality, by considering its discrete form. He did not even think that he had opened the space for numerous researches whose results will be far-reaching and fruitful.

Shortly after discovering the discrete form, the integral form of the Hilbert inequality was also established, as well as the generalization for the case of conjugate exponents. During subsequent decades, the Hilbert inequality was also generalized in many different ways by some famous authors. Nowadays, more than a century after Hilbert's discovery, this problem area is still of interest and provides some possibilities for further generalizations.

In this chapter we present some basic generalizations of the Hilbert inequality. After the short historical overview, we expose a recent important generalization, which provides a unified treatment to this inequality with conjugate exponents. In particular, in that result the integrals are taken with σ-finite measures, which includes both integral and discrete case.

The above mentioned main result is then applied to homogeneous functions, which yields numerous interesting examples. Also, the consideration of such examples in particular settings yields numerous results, previously known from the literature. Moreover, all results presented in two-dimensional case can naturally be extended to a multidimensional case.

Finally, numerous inequalities in this chapter include the corresponding constant factor on their right-hand sides. By the classical Hilbert inequality such constant factor was the best possible in the sense that it cannot be replaced with the smaller constant so that the resulting inequality still remains valid. We shall also present here some recent results which include such best possible constant factors.

1.1 Historical overview

We begin this overview with a bilinear form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n},$$

associated to sequences of real numbers $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, which was first studied by D. Hilbert at the end of the nineteenth century. Hilbert discovered a natural upper bound of this double series and laid the foundations for the theory that will follow. Thus, we present here some basic theorems which arose immediately from Hilbert's considerations.

Theorem 1.1 *Let p and q be mutually conjugate exponents, that is, $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be any two sequences of non-negative real numbers such that $0<\sum_{m=1}^{\infty}a_m^p<\infty$ and $0<\sum_{n=1}^{\infty}b_n^q<\infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{m=1}^{\infty}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}b_n^q\right]^{\frac{1}{q}}. \tag{1.1}$$

The integral form of the previous theorem reads as follows:

Theorem 1.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $f,g:\mathbb{R}_+\to\mathbb{R}$ be any two non-negative Lebesgue measurable functions such that $0<\int_0^\infty f^p(x)dx<\infty$ and $0<\int_0^\infty g^q(y)dy<\infty$. Then*

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}}. \tag{1.2}$$

Remark 1.1 Suppose that p and q are mutually conjugate parameters, i.e. $\frac{1}{p}+\frac{1}{q}=1$, and let $p>1$. Then it follows that $q>1$. On the other hand, if $0<p<1$, then $q<0$, and analogously, if $0<q<1$, then $p<0$.

As we see, in the above theorems the same constant factor appears on the right-hand sides of both inequalities. It was proved by Hardy and Riesz that this constant factor was the best possible.

Theorem 1.3 *The constant $\pi/\sin(\pi/p)$ appearing in (1.1) and (1.2) is the best possible.*

The previous three results are taken from the classical monograph [33], in a slightly altered form. The case of $p = q = 2$ in Theorem 1.1 was first proved by Hilbert in his lectures about integral equations. The lack of that old proof consisted in the fact that Hilbert didn't know to determine the optimal constant factor π. That drawback was removed by Shur in 1911, who also proved the integral version of the inequality. The extensions to arbitrary pair of positive mutually conjugate exponents are due to G.H. Hardy and M. Riesz.

Some other proofs, as well as various generalizations are due to the following mathematicians: L. Fejér, E. Francis, G. H. Hardy, J. Littlewood, H. Mulholland, P. Owen, G. Pólya, F. Riesz, M. Riesz, I. Schur, G. Szegö. Nevertheless, the inequalities (1.1) and (1.2) remained known as the discrete and the integral Hilbert inequalities. For more details about the initial development of the Hilbert inequality the reader is referred to [33, Chapter 9]. It should be noticed here that generalizations of inequalities (1.1) and (1.2) will be referred to as the Hilbert-type inequalities.

However, we provide two more results from the 1920s, which will also play an important role in further investigations. Namely, Hardy, Littlewood and Pólya noted that to every Hilbert-type inequality one can assign its equivalent form, in the sense that one implies another and vice versa. For example, the equivalent form assigned to inequality (1.1) is contained in the following theorem.

Theorem 1.4 *Let $p > 1$ and let $(a_m)_{m\in\mathbb{N}}$ be the sequence of non-negative real numbers such that $0 < \sum_{m=1}^{\infty} a_m^p < \infty$. Then*

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \sum_{m=1}^{\infty} a_m^p. \tag{1.3}$$

Obviously, the integral analogue of inequality (1.3) is analogous, with the sum replaced with the integral, and a sequence with a non-negative real function. Such inequalities, derived from the Hilbert-type inequalities will be referred to as the Hardy-Hilbert-type inequalities. Moreover, the Hilbert-type and the Hardy-Hilbert-type inequalities will sometimes simply be referred to as the Hilbert-type inequalities.

Already at that time, the sharper version of inequality (1.1) was also known. That result is presented in the following theorem.

Theorem 1.5 *Under the same assumptions as in Theorem* 1.1, *we have*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\frac{\pi}{p}}\left[\sum_{m=0}^{\infty} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=0}^{\infty} b_n^q\right]^{\frac{1}{q}}. \tag{1.4}$$

Inequalities (1.1) and (1.4) are known in the literature as "the Hilbert double series theorems".

These theorems were inspiration to numerous mathematicians. During the 20th century numerous proofs, generalizations and applications of the Hilbert inequality were discovered and it would be impossible to count them here.

Nowadays, more than a century after the discovery of the Hilbert inequality, this research area is still interesting to numerous authors. As an illustration, we indicate here some generalizations obtained in the last ten years. One of the possible extensions arises from studying various kernels. Namely, in presented results such kernel was the function $K(x,y) = (x+y)^{-1}$. In 1998, considering the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, B. Yang was the first one to have included the well-known Beta function into the study of Hilbert-type inequalities (see [138]). Recall that the Beta function is an integral function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \qquad a,b > 0. \tag{1.5}$$

For example, in [152] one can find the following result in the integral form.

Theorem 1.6 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s > 2-\min\{p,q\}$. If $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable functions such that $0<\int_0^\infty x^{1-s}f^p(x)dx<\infty$ and $0<\int_0^\infty y^{1-s}g^q(y)dy<\infty$, then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy\\ &< B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)\left[\int_0^\infty x^{1-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&\int_0^\infty y^{(s-1)(p-1)}\left[\int_0^\infty \frac{f(x)}{(x+y)^s}dx\right]^p dy\\ &< B^p\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)\int_0^\infty x^{1-s}f^p(x)dx.\end{aligned} \tag{1.7}$$

Moreover, these two inequalities are equivalent and include the best possible constant factors on their right-hand sides.

The multidimensional extension of inequality (1.6), involving the usual Gamma function, has also been derived in the above mentioned paper [152]. Recall that the Gamma function is an integral

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt, \qquad a>0. \tag{1.8}$$

Theorem 1.7 *Suppose that p_i are mutually conjugate exponents, i.e. $\sum_{i=1}^n \frac{1}{p_i}=1$, $p_i>1$, $i=1,2,\ldots,n$, and let $s>n-\min_{1\le i\le n}\{p_i\}$. If $f_i:\mathbb{R}_+\to\mathbb{R}$, $i=1,2,\ldots,n$, are non-negative measurable functions satisfying $0<\int_0^\infty {x_i}^{n-1-s}f_i^{p_i}(x_i)dx_i<\infty$, then*

$$\begin{aligned}&\int_0^\infty\cdots\int_0^\infty \frac{\prod_{i=1}^n f_i(x_i)}{(\sum_{i=1}^n x_i)^s}dx_1dx_2\ldots dx_n\\ &< \frac{1}{\Gamma(s)}\prod_{i=1}^n \Gamma\left(\frac{p_i+s-n}{p_i}\right)\left[\int_0^\infty {x_i}^{n-1-s}f_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}.\end{aligned} \tag{1.9}$$

Remark 1.2 The above notation for the Beta and the Gamma function will be used throughout the book. The basic relationship between the Beta and the Gamma functions is given by

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a,b>0, \tag{1.10}$$

and this formula will often be exploited. For more details about the Beta and the Gama functions, as well as about their meromorphic extensions to the set of complex numbers, the reader is referred to [1].

It is interesting that the n-dimensional inequality (1.9) also posses its equivalent form, which will be discussed in this chapter.

On the other hand, another possible generalization of the presented results is the investigation of the inequalities of the same type, but where the integrals are taken over a bounded interval in $\mathbb{R}_+$. Guided by that idea, K. Jichang and T. Rassias [42], obtained the following result.

Theorem 1.8 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous symmetric function of degree $-s$, where $\max\{\frac{1}{p},\frac{1}{q}\}<s$. If $K(1,y)$ is strictly decreasing in y and $f,g:\mathbb{R}_+\to\mathbb{R}$ are non-negative measurable functions, then*

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy$$
$$\leq\left[\int_a^b\Big(I(q)-\varphi(q,x)\Big)x^{1-s}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b\Big(I(p)-\varphi(p,y)\Big)y^{1-s}g^q(y)dy\right]^{\frac{1}{q}}, \tag{1.11}$$

where

$$\varphi(r,x)=\left(\frac{a}{x}\right)^{1-\frac{1}{r}}\int_0^1 K(1,u)u^{-\frac{1}{r}}du+\left(\frac{x}{b}\right)^{s+\frac{1}{r}-1}\int_0^1 K(1,u)u^{s+\frac{1}{r}-2}du,$$

$I(r)=\int_0^\infty K(1,u)u^{-\frac{1}{r}}du$, $r\in\{p,q\}$, and $0\leq a<b\leq\infty$.

In the next section, the integrals will be taken over more general sets.

1.2 A unified treatment of Hilbert-type inequalities with conjugate exponents

In the previous historical overview we have seen the classical Hilbert inequality in both discrete and integral case. Moreover, throughout years numerous extensions of these inequalities were derived. However, all these results were given in either integral form, with respect to the Lebesgue measure, or in the discrete form.

The main objective of this section is to present a general result which unifies the integral and discrete cases. This can be done by observing a more general integral. Namely, the classical Hilbert inequality is a consequence of the Hölder inequality and the Fubini theorem. In general, the Fubini theorem holds for the integrals with σ-finite measures, therefore, such measures will be considered.

The most important examples of σ-finite measures are the Lebesgue measure and the counting measure. The Lebesgue measure yields the classical integral case, while the counting measure provides the discrete case.

Further, it is well-known that if one of the mutually conjugate exponents in the Hölder inequality is negative, then the sign of the inequality is reversed (see [103]). Hence, we shall also be concerned with the Hilbert-type inequalities with the reversed sign of inequality. Such inequalities will be referred to as the reverse inequalities.

Now we present the most general form of the Hilbert inequality in the setting described above. It should be noticed here that we suppose that all integrals converge, and such types of conditions will often be omitted. Moreover, integrals will be taken over a general measure space. Results that follow are provided in two equivalent forms: the Hilbert and the Hardy-Hilbert forms.

Theorem 1.9 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let Ω be a measure space with positive σ-finite measures μ_1 and μ_2. Let $K:\Omega\times\Omega\to\mathbb{R}$ and $\varphi,\psi:\Omega\to\mathbb{R}$ be non-negative measurable functions. If the functions F and G are defined by $F(x)=\int_\Omega K(x,y)\psi^{-p}(y)d\mu_2(y)$ and $G(y)=\int_\Omega K(x,y)\varphi^{-q}(x)d\mu_1(x)$, then for all non-negative measurable functions f and g on Ω the inequalities*

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)$$
$$\leq\left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}} \tag{1.12}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)$$
$$\leq\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x) \tag{1.13}$$

hold and are equivalent.

If $0<p<1$, then the reverse inequalities in (1.12) *and* (1.13) *are valid, as well as the inequality*

$$\int_\Omega F^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)$$
$$\leq\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y). \tag{1.14}$$

Proof. The left-hand side of inequality (1.12) can be rewritten in the following form:

$$\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)=\int_\Omega\int_\Omega K(x,y)f(x)\frac{\varphi(x)}{\psi(y)}g(y)\frac{\psi(y)}{\varphi(x)}d\mu_1(x)d\mu_2(y).$$

Now, applying the Hölder inequality to the above relation yields

$$\begin{aligned}
&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&\le\left[\int_\Omega\int_\Omega K(x,y)f^p(x)\frac{\varphi^p(x)}{\psi^p(y)}d\mu_1(x)d\mu_2(y)\right]^{\frac1p}\\
&\quad\times\left[\int_\Omega\int_\Omega K(x,y)g^q(y)\frac{\psi^q(y)}{\varphi^q(x)}d\mu_1(x)d\mu_2(y)\right]^{\frac1q}.
\end{aligned}$$

Finally, using the Fubini theorem and definitions of functions F and G we obtain (1.12).

Now, we are going to show the equivalence of inequalities (1.12) and (1.13). For that sake, suppose that inequality (1.12) holds. Defining the function g by

$$g(y)=G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^{p-1},$$

taking into account that $\frac1p+\frac1q=1$, and using (1.12), we have

$$\begin{aligned}
&\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\
&=\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&\le\left[\int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac1p}\left[\int_\Omega\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac1q}\\
&=\left[\int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac1p}\\
&\quad\times\left[\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\right]^{\frac1q},
\end{aligned}$$

that is, we get (1.13).

On the other hand, suppose that inequality (1.13) holds. In that case, another use of the Hölder inequality yields

$$\begin{aligned}
&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\
&=\int_\Omega\left[\psi^{-1}(y)G^{-\frac1q}(y)\int_\Omega K(x,y)f(x)d\mu_1(x)\right]\psi(y)G^{\frac1q}(y)g(y)d\mu_2(y)\\
&\le\left[\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)d\mu_1(x)\right)^p d\mu_2(y)\right]^{\frac1p}\\
&\quad\times\left[\int_\Omega\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac1q}\\
&\le\left[\int_\Omega\varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac1p}\left[\int_\Omega\psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac1q},
\end{aligned}$$

which implies (1.12). Therefore, inequalities (1.12) and (1.13) are equivalent.

The reverse inequalities, as well as their equivalence, are derived in the same way by virtue of the reverse Hölder inequality. □

Remark 1.3 The equality in the previous theorem is possible if and only if it holds in the Hölder inequality, that is, if

$$\left[f(x)\frac{\varphi(x)}{\psi(y)}\right]^p = C\left[g(y)\frac{\psi(y)}{\varphi(x)}\right]^q, \quad \text{a. e. on } \Omega,$$

where C is a positive constant. In that case we have

$$f(x) = C_1\varphi^{-q}(x) \quad \text{and} \quad g(y) = C_2\psi^{-p}(y) \quad \text{a. e. on } \Omega, \tag{1.15}$$

for some constants C_1 and C_2, which is possible if and only if

$$\int_\Omega F(x)\varphi^{-q}(x)d\mu_1(x) < \infty \quad \text{and} \quad \int_\Omega G(y)\psi^{-p}(y)d\mu_2(y) < \infty. \tag{1.16}$$

Otherwise, the inequalities in Theorem 1.9 are strict.

In some applications of the previous theorem it will be more convenient to bound the functions $F(x)$ and $G(y)$. Of course, such result follows immediately from Theorem 1.9.

Theorem 1.10 *Suppose that the assumptions as in Theorem* 1.9 *are fulfilled and let* $F_1, G_1 : \Omega \to \mathbb{R}$ *be non-negative measurable functions such that* $F(x) \leq F_1(x)$ *and* $G(y) \leq G_1(y)$*, a. e. on* Ω*. Then the inequalities*

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\leq \left[\int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G_1(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}&\int_\Omega {G_1}^{1-p}(y)\psi^{-p}(y)\left[\int_\Omega K(x,y)f(x)d\mu_1(x)\right]^p d\mu_2(y)\\ &\leq \int_\Omega \varphi^p(x)F_1(x)f^p(x)d\mu_1(x)\end{aligned} \tag{1.18}$$

hold and are equivalent.

If $0 < p < 1$, $F(x) \geq F_1(x)$, *and* $G(y) \leq G_1(y)$, *then the reverse inequalities in* (1.17) *and* (1.18) *hold, as well as the inequality*

$$\begin{aligned}&\int_\Omega {F_1}^{1-q}(x)\varphi^{-q}(x)\left[\int_\Omega K(x,y)g(y)d\mu_2(y)\right]^q d\mu_1(x)\\ &\leq \int_\Omega \psi^q(y)G_1(y)g^q(y)d\mu_2(y).\end{aligned} \tag{1.19}$$

The reverse inequalities are also equivalent.

Remark 1.4 The general Hilbert-type inequalities presented in this section have been obtained by M. Krnić and J. Pečarić in [53].

1.3 Applications to homogeneous kernels

Theorem 1.9 from the previous section has unified the classical integral and discrete cases of the Hilbert inequality. In order to approach to some well-known results from the literature, we study here some particular choices of kernels and weight functions.

In this section we consider homogeneous kernels of negative degree of homogeneity, equipped with some additional properties. Recall that a function $K:\Omega\times\Omega\to\mathbb{R}$ is said to be homogeneous of degree $-s$, $s>0$, if $K(tx,ty)=t^{-s}K(x,y)$ for every $x,y\in\Omega$ and $t\in\mathbb{R}$ such that tx, $ty\in\Omega$. In addition, for such homogeneous function we define $k(\alpha)$ as

$$k(\alpha)=\int_0^\infty K(1,u)u^{-\alpha}du, \tag{1.20}$$

provided that the above integral converges for $1-s<\alpha<1$.

We study here the integral case, that is, the Lebesgue integral. The integrals are taken over an arbitrary interval of non-negative real numbers, i.e. $(a,b)\subseteq\mathbb{R}_+$, $0\le a<b\le\infty$, and the weight functions are chosen to be power functions.

Theorem 1.11 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a,b)\times(a,b)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ the inequalities*

$$\begin{aligned}&\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy\\ &\le\left[\int_a^b\big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\\ &\quad\times\left[\int_a^b\big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}\end{aligned} \tag{1.21}$$

and

$$\begin{aligned}&\int_a^b\big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)^{1-p}y^{(p-1)(s-1)+p(A_1-A_2)}\\ &\quad\times\left[\int_a^b K(x,y)f(x)dx\right]^p dy\\ &\le\int_a^b\big(k(pA_2)-\varphi_1(pA_2,x)\big)x^{1-s+p(A_1-A_2)}f^p(x)dx\end{aligned} \tag{1.22}$$

hold and are equivalent, where

$$\varphi_1(\alpha,x)=\left(\frac{a}{x}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\left(\frac{x}{b}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\varphi_2(\alpha,y)=\left(\frac{a}{y}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y}{b}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, and $K(x,y)$ is strictly decreasing in x and strictly increasing in y, then the reverse inequalities in (1.21) and (1.22) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_a^\infty \big(k(pA_2)-\varphi_1(pA_2,x)\big)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_a^\infty K(x,y)g(y)dy\right]^q dx$$
$$\leq \int_a^\infty \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Moreover, if $0<p<1$, $a=0$, and $K(x,y)$ is strictly increasing in x and strictly decreasing in y, then the reverse inequalities in (1.21) and (1.22) hold for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_0^b \big(k(pA_2)-\varphi_1(pA_2,x)\big)^{1-q}x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^b K(x,y)g(y)dy\right]^q dx$$
$$\leq \int_0^b \big(k(2-s-qA_1)-\varphi_2(2-s-qA_1,y)\big)y^{1-s+q(A_2-A_1)}g(y)^q dy.$$

Proof. We only prove inequality (1.21). After substituting the power functions $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$ in (1.12), the homogeneity of the kernel K and the substitution $u=\frac{y}{x}$ yield the following relation:

$$\int_a^b\int_a^b K(x,y)f(x)g(y)dxdy$$
$$\leq \left[\int_a^b x^{1-s+p(A_1-A_2)}\left(\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du\right)f^p(x)dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b y^{1-s+q(A_2-A_1)}\left(\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du\right)g^q(y)dy\right]^{\frac{1}{q}}.$$

In addition, considering the function $l(y)=y^{\alpha-1}\int_0^y K(1,u)u^{-\alpha}du$, $\alpha<1$, the integration by parts yields equality

$$l'(y)=y^{\alpha-2}\int_0^y u^{1-\alpha}\frac{\partial K(1,u)}{\partial u}du.$$

Since the kernel K is strictly decreasing in both variables, it follows that $l'(y)<0$, $y\in\mathbb{R}_+$, that is, l is strictly decreasing on $\mathbb{R}_+$.

On the other hand, since

$$\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du = \int_0^\infty K(1,u)u^{-pA_2}du-\int_0^{\frac{a}{x}} K(1,u)u^{-pA_2}du$$
$$-\int_0^{\frac{x}{b}} K(u,1)u^{pA_2+s-2}du,$$

and due to the fact that l is strictly decreasing on $\mathbb{R}_+$, we obtain the estimate

$$\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-pA_2}du \leq k(pA_2) - \varphi_1(pA_2, x)$$

and similarly,

$$\int_{\frac{y}{b}}^{\frac{y}{a}} K(1,u)u^{qA_1+s-2}du \leq k(2-s-qA_1) - \varphi_2(2-s-qA_1, y),$$

so the result follows from Theorem 1.9. Note also that the intervals defining the parameters A_1 and A_2 arise from the assumption on the convergence of integral (1.20). $\square$

Remark 1.5 If the kernel K in the previous theorem is a symmetric function, then $k(2-s-qA_1) = k(qA_1)$. Then, setting $A_1 = A_2 = \frac{1}{pq}$ in Theorem 1.11, provided that $\max\{\frac{1}{p}, \frac{1}{q}\} < s$, we get Theorem 1.8 (see also [42]).

Remark 1.6 In order to justify the convergence interval $(1-s, 1)$ for the integral $k(\alpha)$ defined by (1.20), observe that the homogeneity of the kernel K implies the following sequence of identities:

$$k(\alpha) = \int_0^\infty K\left(\frac{1}{u}, 1\right) u^{-s-\alpha}du = \int_0^\infty K(u,1)u^{s+\alpha-2}du.$$

On the other hand, assuming that K is strictly decreasing in each argument, K is strictly positive on $\mathbb{R}_+ \times \mathbb{R}_+$. In particular, for $\alpha \geq 1$, monotonicity of K in the second argument and the fact that $K(1,1) > 0$ yield

$$k(\alpha) = \int_0^\infty K(1,u)u^{-\alpha}du \geq \int_0^1 K(1,u)u^{-\alpha}du \geq K(1,1)\int_0^1 u^{-\alpha}du = \infty.$$

Analogous result holds also for $\alpha \leq 1-s$, since

$$\begin{aligned} k(\alpha) &= \int_0^\infty K(u,1)u^{s+\alpha-2}du \geq \int_0^1 K(u,1)u^{s+\alpha-2}du \\ &\geq K(1,1)\int_0^1 u^{s+\alpha-2}du = \infty. \end{aligned}$$

Therefore, the interval $(1-s, 1)$, considered in definition (1.20), covers all arguments α for which $k(\alpha)$ may converge. The same conclusion on convergence of $k(\alpha)$ can be drawn if we consider a function K increasing in each argument and such that $K(1,1) > 0$.

It is interesting to consider a particular case of the previous theorem, that is, when the integrals are taken over the whole set $\mathbb{R}_+$. Then, $a = 0$, $b = \infty$, and we obtain the corresponding inequalities for an arbitrary non-negative homogeneous function of degree $-s$.

Corollary 1.1 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then for all non-negative measurable functions $f,g:\mathbb{R}_+\to\mathbb{R}$ the inequalities*

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \leq L\left[\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \tag{1.23}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy \leq L^p\int_0^\infty x^{1-s+p(A_1-A_2)}f^p(x)dx \tag{1.24}$$

hold and are equivalent, where $L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$.

If $0<p<1$, then the reverse inequalities in (1.23) and (1.24) are valid for every $A_1\in(\frac{1}{q},\frac{1-s}{q})$ and $A_2\in(\frac{1-s}{p},\frac{1}{p})$, as well as the inequality

$$\int_0^\infty x^{(q-1)(s-1)+q(A_2-A_1)}\left[\int_0^\infty K(x,y)g(y)dy\right]^q dx \leq L^q\int_0^\infty y^{1-s+q(A_2-A_1)}g^q(y)dy. \tag{1.25}$$

Inequalities (1.23) and (1.24), as well as their reverse inequalities are equivalent. Moreover, equality in the above relations holds if and only if $f=0$ or $g=0$ a.e. on $\mathbb{R}_+$.

Proof. The proof follows immediately from Theorem 1.11 by substituting $a=0$ and $b=\infty$. Moreover, condition (1.15) gives the nontrivial case of equality in (1.23), while condition (1.16) leads to the divergent integrals. Hence, the observed inequalities are strict, unless $f=0$ or $g=0$ a. e. on $\mathbb{R}_+$. □

In the sequel we consider some generalizations of Theorem 1.11. For example, utilizing the substitution $u=x+\mu$ and $v=y+\mu$, $\mu\geq 0$, we have:

Theorem 1.12 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K:(a+\mu,b+\mu)\times(a+\mu,b+\mu)\to\mathbb{R}$ be a non-negative homogeneous function of degree $-s$, $s>0$, strictly decreasing in both variables. If A_1 and A_2 are real parameters such that $A_1\in(\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$, then* *for all non-negative measurable functions* $f,g:(a,b)\to\mathbb{R}$ *the inequalities*

$$\begin{aligned}
&\int_a^b\int_a^b K(x+\mu,y+\mu)f(x)g(y)dxdy\\
&\le\left[\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\mu)\big)(x+\mu)^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\\
&\quad\times\left[\int_a^b\big(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu)\big)(y+\mu)^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}}
\end{aligned}\tag{1.26}$$

and

$$\begin{aligned}
&\int_a^b(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu))^{1-p}(y+\mu)^{(p-1)(s-1)+p(A_1-A_2)}\\
&\quad\times\left[\int_a^b K(x+\mu,y+\mu)f(x)dx\right]^p dy\\
&\le\int_a^b\big(k(pA_2)-\psi_1(pA_2,x,\mu)\big)(x+\mu)^{1-s+p(A_1-A_2)}f^p(x)dx
\end{aligned}\tag{1.27}$$

hold and are equivalent, where

$$\psi_1(\alpha,x,\lambda)=\left(\frac{a+\lambda}{x+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du+\left(\frac{x+\lambda}{b+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du,$$

$$\psi_2(\alpha,y,\lambda)=\left(\frac{a+\lambda}{y+\lambda}\right)^{s+\alpha-1}\int_0^1 K(u,1)u^{s+\alpha-2}du+\left(\frac{y+\lambda}{b+\lambda}\right)^{1-\alpha}\int_0^1 K(1,u)u^{-\alpha}du.$$

If $0<p<1$, $b=\infty$, *and* $K(x,y)$ *is strictly decreasing in* x *and strictly increasing in* y, *then the reverse inequalities in* (1.26) *and* (1.27) *hold for all* $A_1\in(\frac{1}{q},\frac{1-s}{q})$ *and* $A_2\in(\frac{1-s}{p},\frac{1}{p})$, *as well as the inequality*

$$\begin{aligned}
&\int_a^\infty(k(pA_2)-\psi_1(pA_2,x,\mu))^{1-q}(x+\mu)^{(q-1)(s-1)+q(A_2-A_1)}\\
&\quad\times\left[\int_a^\infty K(x+\mu,y+\mu)g(y)dy\right]^q dx\\
&\le\int_a^\infty(k(2-s-qA_1)-\psi_2(2-s-qA_1,y,\mu))(y+\mu)^{1-s+q(A_2-A_1)}g^q(y)dy.
\end{aligned}\tag{1.28}$$

Moreover, inequalities (1.26) *and* (1.27), *as well as their reverses, are equivalent.*

Remark 1.7 Considering Theorem 1.12 with a symmetric kernel and parameters $A_1=A_2=\frac{2\lambda}{pq}$, provided that $0<1-\frac{2\lambda}{p}<s$, $0<1-\frac{2\lambda}{q}<s$, we obtain the corresponding result from [42].

Remark 1.8 Some other ways of generalizing Theorem 1.11 arise from various substitutions. For example, in [53] the authors also use the substitution $u = Ax^{\alpha}$ and $v = By^{\beta}$, where $A, B, \alpha, \beta > 0$. Such results are here omitted. It should be noticed here that the results in this section are taken from the above mentioned paper [53]. In addition, for some more specific Hilbert-type inequalities with a homogeneous kernel the reader is referred to [164] and [181].

1.4 Examples. The best possible constants

This section is dedicated to Hilbert-type inequalities with some particular homogeneous kernels and weight functions. Numerous interesting examples will be given here. Moreover, the best possible constant factors will be derived in some particular settings.

1.4.1 Integral case

We start with the classical integral case. We are concerned here with Corollary 1.1 from the previous section. It is not hard to see that this corollary covers Theorems 1.2 and 1.6, presented in the historical overview at the beginning of this chapter.

Namely, if $K(x,y) = (x+y)^{-s}$, $s > 0$, then the integral (1.20) is expressed in terms of the Beta function, that is, $k(\alpha) = B(1-\alpha, s+\alpha-1)$. Hence, in this setting the constant factor L on the right-hand sides of inequalities (1.23) and (1.24) takes the form

$$L = B^{\frac{1}{p}}(1-pA_2, s+pA_2-1)B^{\frac{1}{q}}(1-qA_1, s+qA_1-1).$$

Moreover, if $A_1 = A_2 = \frac{2-s}{pq}$, then the above constant coincides with the constant factor on the right-hand side of inequality (1.6). Hence, Corollary 1.1 can be regarded as an extension of both Theorems 1.2 and 1.6.

Further, if $K(x,y) = \max\{x,y\}^{-s}$, $s > 0$, then the above constant L, included in (1.23) and (1.24), reads

$$L = \frac{s}{(1-pA_2)^{\frac{1}{p}}(1-qA_1)^{\frac{1}{q}}(s+pA_2-1)^{\frac{1}{p}}(s+qA_1-1)^{\frac{1}{q}}}.$$

Similarly, for $A_1 = A_2 = \frac{2-s}{pq}$ the above constant factor reduces to

$$L = \frac{pqs}{(p+s-2)(q+s-2)},$$

and the resulting Hilbert-type inequality coincides with the one from [42].

On the other hand, Hilbert-type inequalities in Theorems 1.2 and 1.6, as well as the above mentioned result from [42], include the best possible constant factor.

Our main task is to determine conditions under which the constant factor $L = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1)$ is the best possible in inequalities (1.23) and (1.24). Observe that inequalities

(1.2) and (1.6) include the best possible constant factors without any exponent. Guided by that fact we are going to simplify the constant factor L from Corollary 1.1. More precisely, if we set the condition

$$pA_2 + qA_1 = 2 - s, \tag{1.29}$$

then the constant factor L in Corollary 1.1 reduces to $L = k(pA_2)$.

In the sequel, we are going to show that, under the above condition (1.29) and assuming some weak conditions on the kernel, inequalities in Corollary 1.1 include the best possible constant factors. In order to prove that result we need the following lemma.

Lemma 1.1 *Let p and q be conjugate parameters with $p > 1$. If $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative measurable function such that $K(1,t)$ is bounded on $(0,1)$, then*

$$\int_1^\infty x^{-\varepsilon-1}\left(\int_0^{1/x} K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx = O(1),\ \varepsilon \to 0^+, \tag{1.30}$$

where $A_2 < \frac{1}{p}$.

Proof. Using the assumptions, we have $K(1,t) \le C$ for some $C > 0$ and every $t \in (0,1)$. Let $\varepsilon > 0$ be such that $\varepsilon < pq\left(\frac{1}{p} - A_2\right)$. We have

$$\begin{aligned}\int_1^\infty x^{-1-\varepsilon}\left(\int_0^{1/x} K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx &\le C\int_1^\infty x^{-1-\varepsilon}\left(\int_0^{1/x} t^{-pA_2-\frac{\varepsilon}{q}}dt\right)dx \\ = \frac{C}{1-pA_2-\frac{\varepsilon}{q}}\int_1^\infty x^{pA_2+\frac{\varepsilon}{q}-\varepsilon-2}dx &= \frac{C}{\left(1-pA_2-\frac{\varepsilon}{q}\right)\left(1-pA_2+\frac{\varepsilon}{p}\right)},\end{aligned}$$

wherefrom (1.30) follows. □

Theorem 1.13 *Suppose that the assumptions of Corollary 1.1 are fulfilled. Additionally, if the kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is such that $K(1,t)$ is bounded on $(0,1)$, and if the parameters A_1 and A_2 fulfill condition (1.29), then the constants $L = k(pA_2)$ and $L^p = k^p(pA_2)$ are the best possible in both inequalities (1.23) and (1.24).*

Proof. For this purpose, with $0 < \varepsilon < pq\left(\frac{1}{p} - A_2\right)$, set $f(x) = x^{-qA_1-\frac{\varepsilon}{p}}\chi_{[1,\infty)}(x)$ and $g(y) = y^{-pA_2-\frac{\varepsilon}{q}}\chi_{[1,\infty)}(y)$, where χ_A is the characteristic function of a set A.
Now, suppose that there exists a smaller constant $0 < M < L$ such that inequality (1.23) holds. Let I denote the right-hand side of (1.23). Then,

$$I = M\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(\int_1^\infty y^{-\varepsilon-1}dy\right)^{\frac{1}{q}} = \frac{M}{\varepsilon}. \tag{1.31}$$

Applying respectively the Fubini theorem, substitution $t = \frac{y}{x}$, and Lemma 1.1, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty K(x,y) f(x) g(y)\,dxdy \\
&= \int_1^\infty x^{-qA_1-\frac{\varepsilon}{p}} \left(\int_1^\infty K(x,y) y^{-pA_2-\frac{\varepsilon}{q}} dy \right) dx \\
&= \int_1^\infty x^{-\varepsilon-1} \left(\int_0^\infty K(1,t) t^{-pA_2-\frac{\varepsilon}{q}} dt - \int_0^{x^{-1}} K(1,t) t^{-pA_2-\frac{\varepsilon}{q}} dt \right) dx \\
&= \frac{1}{\varepsilon}\left[k\left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) \right]. \qquad (1.32)
\end{aligned}$$

From (1.23), (1.31), and (1.32) we get

$$k\left(pA_2 + \frac{\varepsilon}{q} \right) + o(1) < M. \qquad (1.33)$$

Now, letting $\varepsilon \to 0^+$, relation (1.33) yields a contradiction with the assumption $M < L = k(pA_2)$.

Finally, equivalence of inequalities (1.23) and (1.24) means that the constant $L^p = k^p(pA_2)$ is also the best possible in (1.24). The proof is now completed. □

As we see, the previous theorem covers the problem of finding the best possible constant factors for a quite weak condition on homogeneous kernel and parameters A_1, A_2 satisfying (1.29). We have already considered the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$. This kernel fulfills the above mentioned condition from Theorem 1.13, hence, the best possible constant in that case takes the form $B(1-pA_2, 1-qA_1)$. Hilbert-type inequalities with this kernel and parameters A_1, A_2 were extensively studied in recent papers [10], [11], [52], [53], and [54].

Some other examples of the best possible constant factors arise from various choices of kernels. For example, considering the kernel $K(x,y) = (x+y+\max\{x,y\})^{-s}$, $s > 0$, the best possible constant factor $k(pA_2)$ from Theorem 1.13 becomes

$$\frac{2^{-s}}{pA_2+s-1} F\left(s, s+pA_2-1; s+pA_2; -1/2\right) + \frac{2^{-s}}{1-pA_2} F\left(s, 1-pA_2; 2-pA_2; -1/2\right),$$

where $F(\alpha,\beta;\gamma;z)$ denotes the Gaussian hypergeometric function, that is,

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-zt)^\alpha} dt,\ \gamma > \beta > 0, z < 1. \qquad (1.34)$$

The above kernel with degree of homogeneity equal to -1 was also discussed in [81].

We conclude this subsection with some particular Hilbert-type inequalities equipped with homogeneous kernels of degree -1, involving the best possible constant factors.

Remark 1.9 Setting $s=1$, $A_1=A_2=\frac{1}{pq}$ in Corollary 1.1, inequalities (1.23) and (1.24) become respectively

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \le k(1/q)\left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}} \tag{1.35}$$

and

$$\int_0^\infty\left[\int_0^\infty K(x,y)f(x)dx\right]^p dy \le k^p(1/q)\int_0^\infty f^p(x)dx. \tag{1.36}$$

The following kernels K are homogenous with bounded $K(1,t)$ on $(0,1)$. For each of these functions we compute constants $L=k(1/q)$ and $L_2=k(1/2)$, that is, when $p=q=2$:

$$K(x,y)=\frac{1}{x+y+\max\{x,y\}},$$
$$L=\frac{1}{2}qF\left(1,\frac{1}{q};1+\frac{1}{q};-\frac{1}{2}\right)+\frac{1}{2}pF\left(1,\frac{1}{p};1+\frac{1}{p};-\frac{1}{2}\right),$$
$$L_2=\sqrt{2}(\pi-2\arctan\sqrt{2});$$
$$K(x,y)=\frac{1}{x+y+\min\{x,y\}},$$
$$L=qF\left(1,\frac{1}{q};1+\frac{1}{q};-2\right)+pF\left(1,\frac{1}{p};1+\frac{1}{p};-2\right),L_2=2\sqrt{2}\arctan\sqrt{2};$$
$$K(x,y)=\frac{1}{|x-y|+\max\{x,y\}},$$
$$L=\frac{1}{2}qF\left(1,\frac{1}{q};1+\frac{1}{q};\frac{1}{2}\right)+\frac{1}{2}pF\left(1,\frac{1}{p};1+\frac{1}{p};\frac{1}{2}\right),L_2=2\operatorname{artanh}\frac{1}{\sqrt{2}};$$
$$K(x,y)=\frac{1}{x+y-\min\{x,y\}}=\frac{1}{\max\{x,y\}},L=pq,L_2=4;$$
$$K(x,y)=\frac{1}{x+y+\frac{2}{\frac{1}{x}+\frac{1}{y}}},L_2=\sqrt{\frac{2}{3}}\pi;$$
$$K(x,y)=\frac{1}{x+y-\frac{2}{\frac{1}{x}+\frac{1}{y}}},L=\frac{\pi}{2}\left(\frac{1}{\cos\frac{\pi}{2p}}+\frac{1}{\cos\frac{\pi}{2q}}\right),L_2=\pi\sqrt{2};$$
$$K(x,y)=\frac{1}{x+y+\sqrt{xy}},L_2=\frac{4\pi}{3\sqrt{3}};$$
$$K(x,y)=\frac{1}{x+y-\sqrt{xy}},L_2=\frac{8\pi}{3\sqrt{3}};$$
$$K(x,y)=\frac{x^{\lambda-1}+y^{\lambda-1}}{x^\lambda+y^\lambda},L=\frac{\pi}{\lambda}\left(\frac{1}{\sin\frac{\pi}{\lambda p}}+\frac{1}{\sin\frac{\pi}{\lambda q}}\right),\lambda\ge 1;$$
$$K(x,y)=\frac{x^{\lambda-1}-y^{\lambda-1}}{x^\lambda-y^\lambda},L=\frac{\pi}{\lambda}\left(\cot\frac{\pi}{\lambda p}+\cot\frac{\pi}{\lambda q}\right),\lambda>1;$$

$$K(x,y) = \frac{\log y - \log x}{y-x}, L = \frac{\pi^2}{\left(\sin\frac{\pi}{p}\right)^2}, L_2 = \pi^2.$$

Since parameters $s=1, A_1 = A_2 = \frac{1}{pq}$ fulfill condition $pA_2 + qA_1 = 2-s$, all these constant factors are the best possible in both inequalities (1.35) and (1.36).

1.4.2 Discrete case

Discrete case of the Hilbert inequality is more complicated than the integral one. Namely, in order to obtain discrete forms of the corresponding integral inequalities, it is necessary to do some further estimates, which requires some additional conditions.

In Section 1.1 we encountered the Hilbert double series theorems, those were inequalities (1.1) and (1.4). Moreover, the corresponding equivalent form assigned to (1.1) is inequality (1.3), while the equivalent form assigned to (1.4) was derived in [142].

Recently, M. Krnić and J. Pečarić (see [52]), obtained the following discrete version of the Hilbert inequality with conjugate parameters $p,q>1$ and real parameters $A,B,\alpha,\beta,s>0$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(Am^\alpha + Bn^\beta)^s} \le M\left[\sum_{m=1}^{\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)} a_m^p\right]^{\frac{1}{p}} \times\left[\sum_{n=1}^{\infty} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)} b_n^q\right]^{\frac{1}{q}}, \quad (1.37)$$

where $(a_m)_{m\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are non-negative real sequences, $A_1 \in \left(\max\{\frac{1-s}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q}\right), A_2 \in \left(\max\{\frac{1-s}{p}, \frac{\beta-1}{\beta p}\}, \frac{1}{p}\right)$ and

$$M = \alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{\frac{2-s}{p}+A_1-A_2-1}B^{\frac{2-s}{q}+A_2-A_1-1} \times B^{\frac{1}{p}}(1-pA_2, s-1+pA_2)B^{\frac{1}{q}}(1-qA_1, s-1+qA_1).$$

The equivalent form that corresponds to (1.37) is also derived in [52].

Similarly, considering parameters ϕ, ψ, and λ, such that $0<\phi,\psi\le 1$ and $\phi+\psi=\lambda$, B. Yang [162], obtained the following pair of equivalent inequalities

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log\left(\frac{m}{n}\right)a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda\sin\left(\frac{\pi\phi}{\lambda}\right)}\right]^2 \left[\sum_{n=1}^{\infty} n^{p(1-\phi)-1}a_n^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\psi)-1}b_n^q\right]^{\frac{1}{q}} \quad (1.38)$$

and

$$\sum_{n=1}^{\infty} n^{p\psi-1}\left[\sum_{m=1}^{\infty}\frac{\log\left(\frac{m}{n}\right)a_m b_n}{m^\lambda - n^\lambda}\right]^p < \left[\frac{\pi}{\lambda\sin\left(\frac{\pi\phi}{\lambda}\right)}\right]^{2p}\sum_{n=1}^{\infty} n^{p(1-\phi)-1}a_n^p, \quad (1.39)$$

which hold for all non-negative conjugate exponents and non-negative sequences fulfilling $0 < \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q < \infty$. Moreover, the constant factors included in the right-hand sides of inequalities are the best possible.

On the other hand, B. Yang and T. M. Rassias [152] (see also [149]), studied the kernel expressed in terms of the logarithm function. They obtained the following pair of equivalent inequalities,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\log mn} < \frac{\pi}{\sin\pi/p}\left[\sum_{n=2}^{\infty} n^{p-1}a_n^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty} n^{q-1}b_n^q\right]^{\frac{1}{q}} \tag{1.40}$$

and

$$\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{a_m}{\log mn}\right)^p < \left[\frac{\pi}{\sin\pi/p}\right]^p \sum_{n=2}^{\infty} n^{p-1}a_n^p, \tag{1.41}$$

which hold for non-negative conjugate exponents and non-negative sequences such that $0 < \sum_{n=2}^{\infty} n^{p-1} a_n^p < \infty$ and $0 < \sum_{n=2}^{\infty} n^{q-1} b_n^q < \infty$. Moreover, the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$, on the right-hand sides of inequalities (1.40) and (1.41), are the best possible. Observe that the above inequality (1.40) for $p = q = 2$ is also known as the Mulholland inequality.

Clearly, the kernel involved in the previous two inequalities, as well as in (1.37) is non-homogeneous, while the kernel in (1.38) and (1.39) is homogeneous.

However, utilizing suitable substitutions, these non-homogeneous kernels can also be interpreted as the homogeneous ones. Thus, in the sequel we provide discrete forms of Hilbert-type inequalities with a general homogeneous kernel. The same conditions as in the integral case are assumed on the convergence of the integral $k(\alpha)$, defined by (1.20).

The following result contains discrete Hilbert-type inequalities for a homogeneous kernel in both equivalent forms. Discrete weight functions involve here differentiable real functions. In addition, for the reader's convenience, we introduce here the following notation: $H(r)$, $r > 0$, denotes the set of all non-negative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(i) u is strictly increasing on $\mathbb{R}_+$ and there exists $x_0 \in \mathbb{R}_+$ such that $u(x_0) = 1$,

(ii) $\lim_{x\to\infty} u(x) = \infty$, $\frac{u'(x)}{[u(x)]^r}$ is decreasing on $\mathbb{R}_+$.

Theorem 1.14 *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, *and let* $s > 0$. *Further, suppose that* $A_1 \in \left(\max\{\frac{1-s}{q}, 0\}, \frac{1}{q}\right)$, $A_2 \in \left(\max\{\frac{1-s}{p}, 0\}, \frac{1}{p}\right)$, $u \in H(qA_1)$ *and* $v \in H(pA_2)$. *If* $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is a non-negative homogeneous function of degree* $-s$, *strictly decreasing* *in each argument, then the inequalities*

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&K(u(m),v(n))a_m b_n\\ &\leq L\left[\sum_{m=1}^{\infty}[u(m)]^{1-s+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=1}^{\infty}[v(n)]^{1-s+q(A_2-A_1)}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}}\end{aligned}\tag{1.42}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}[v(n)]^{(s-1)(p-1)+p(A_1-A_2)}v'(n)&\left[\sum_{m=1}^{\infty}K(u(m),v(n))a_m\right]^p\\ &\leq L^p\sum_{m=1}^{\infty}[u(m)]^{(1-s)+p(A_1-A_2)}[u'(m)]^{1-p}a_m^p\end{aligned}\tag{1.43}$$

hold for all non-negative sequences $(a_m)_{m\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, *where*

$$L=k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(2-s-qA_1).\tag{1.44}$$

Moreover, inequalities (1.42) *and* (1.43) *are equivalent.*

Proof. Rewrite inequality (1.12) for the counting measure on $\mathbb{N}$, $(\varphi\circ u)(m)=[u(m)]^{A_1}[u'(m)]^{-\frac{1}{q}}$, $(\psi\circ v)(n)=[v(n)]^{A_2}[v'(n)]^{-\frac{1}{p}}$, and the sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Clearly, these substitutions are well-defined, since u and v are injective functions. Thus, in this setting we have

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&K(u(m),v(n))a_m b_n\\ &\leq\left[\sum_{m=1}^{\infty}[u(m)]^{pA_1}[u'(m)]^{1-p}(F\circ u)(m)a_m^p\right]^{\frac{1}{p}}\\ &\quad\times\left[\sum_{n=1}^{\infty}[v(n)]^{qA_2}[v'(n)]^{1-q}(G\circ v)(n)b_n^q\right]^{\frac{1}{q}},\end{aligned}\tag{1.45}$$

where

$$(F\circ u)(m)=\sum_{n=1}^{\infty}\frac{K(u(m),v(n))v'(n)}{[v(n)]^{pA_2}}$$

and

$$(G\circ v)(n)=\sum_{m=1}^{\infty}\frac{K(u(m),v(n))u'(m)}{[u(m)]^{qA_1}}.$$

Now, since the kernel K is strictly decreasing in each argument and $u \in H(qA_1)$, $v \in H(pA_2)$, it follows that $F \circ u$ and $G \circ v$ are strictly decreasing. Hence, we have

$$(F \circ u)(m) < \int_0^\infty \frac{K(u(m), v(y))}{[v(y)]^{pA_2}} v'(y)dy, \tag{1.46}$$

since the left-hand side of this inequality is obviously the lower Darboux sum for the integral on the right-hand side of inequality. Further, utilizing substitution $v(y) = tu(m)$ and homogeneity of the kernel K, we have

$$\int_0^\infty \frac{K(u(m), v(y))}{[v(y)]^{pA_2}} v'(y)dy = [u(m)]^{1-s-pA_2} \int_0^\infty K(1,t)t^{-pA_2}dt,$$

so by virtue of (1.20) and (1.46) we get

$$(F \circ u)(m) < [u(m)]^{1-s-pA_2} k(pA_2). \tag{1.47}$$

By the similar arguments as for the function $F \circ u$, we obtain

$$\begin{aligned}
(G \circ v)(m) &< \int_0^\infty \frac{K(u(x), v(n))}{[u(x)]^{qA_1}} u'(x)dx \\
&= [v(n)]^{1-s-qA_1} \int_0^\infty K(t,1)t^{-qA_1}dt \\
&= [v(n)]^{1-s-qA_1} \int_0^\infty K(1,t)t^{-2+s+qA_1}dt \\
&= [v(n)]^{1-s-qA_1} k(2-s-qA_1).
\end{aligned} \tag{1.48}$$

Finally, relations (1.45), (1.47), and (1.48) imply inequality (1.42).

On the other hand, if we rewrite inequality (1.13) with the counting measure on $\mathbb{N}$ and the same functions as in the proof of inequality (1.42), after using estimates (1.47) and (1.48), we also obtain (1.43). □

Clearly, Theorem 1.14 covers discrete Hilbert and Hardy-Hilbert-type inequalities with homogeneous kernels, decreasing in both arguments.

Remark 1.10 Suppose $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are non-negative real sequences, not identically equal to trivial zero sequence. Then, according to estimates (1.47) and (1.48), it follows that inequalities (1.42) and (1.43) are sharp. In other words, equalities in (1.42) and (1.43) hold if and only if $a_m \equiv 0$ or $b_n \equiv 0$.

Remark 1.11 If the homogeneous kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a symmetric function, that is, $K(x,y) = K(y,x)$, for all $x, y \in \mathbb{R}_+$, then the constant L, defined by (1.44), simplifies to $L = k^{\frac{1}{p}}(pA_2)k^{\frac{1}{q}}(qA_1)$.

As emphasized above, inequalities (1.42) and (1.43) are sharp if the sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are not identically equal to the zero sequence. Therefore, it is interesting to consider the problem of finding the best possible constant factors for inequalities (1.42) and (1.43).

The main idea in obtaining the best possible constant factor is a reduction of constant defined by (1.44) to the form without exponents, which was already considered in the integral case. Thus, the parameters A_1 and A_2 fulfill (1.29), that is, $pA_2 + qA_1 = 2 - s$, which implies that $k(pA_2) = k(2 - s - qA_1)$. In such a way, the constant factor L from Theorem 1.14 becomes

$$L^* = k(pA_2). \tag{1.49}$$

Moreover, under assumption (1.29), inequalities (1.42) and (1.43) respectively read

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(u(m),v(n))a_m b_n \le L^* \left[\sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p\right]^{\frac{1}{p}} \times \left[\sum_{n=1}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}b_n^q\right]^{\frac{1}{q}} \tag{1.50}$$

and

$$\sum_{n=1}^{\infty}[v(n)]^{(p-1)(1-pqA_2)}v'(n)\left[\sum_{m=1}^{\infty}K(u(m),v(n))a_m\right]^p \le (L^*)^p \sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}a_m^p. \tag{1.51}$$

The following theorem shows that the constants on the right-hand sides of inequalities (1.50) and (1.51) are the best possible.

Theorem 1.15 *Suppose that parameters p, q, s, A_1, A_2, and the functions $u, v : \mathbb{R}_+ \to \mathbb{R}$, $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are defined as in the statement of Theorem* 1.14*. If parameters A_1 and A_2 fulfill condition $pA_2 + qA_1 = 2 - s$, then the constant factors L^* and $(L^*)^p$ are the best possible in inequalities* (1.50) *and* (1.51).

Proof. It is enough to show that L^* is the best possible constant factor in inequality (1.50), since (1.50) and (1.51) are equivalent. For this purpose, we consider sequences $\widetilde{a}_m = [u(m)]^{-qA_1-\frac{\varepsilon}{p}}u'(m)$ and $\widetilde{b}_n = [v(n)]^{-pA_2-\frac{\varepsilon}{q}}v'(n)$, where $\varepsilon > 0$ is sufficiently small number. Since $u \in H(qA_1)$ we may assume that u is strictly increasing in $\mathbb{R}_+$, and there exists $x_0 \in \mathbb{R}_+$ such that $u(x_0) = 1$. Therefore, considering integral sums, we have

$$\begin{aligned}
\frac{1}{\varepsilon} &= \int_1^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)] < \sum_{m=1}^{\infty}[u(m)]^{-1-\varepsilon}u'(m) \\
&= \sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p \\
&< \zeta(1) + \int_1^{\infty}[u(x)]^{-1-\varepsilon}d[u(x)] = \zeta(1) + \frac{1}{\varepsilon},
\end{aligned}$$

where the function ζ is defined by $\zeta(x) = [u(x)]^{-1-\varepsilon}u'(x)$. Hence, we have

$$\sum_{m=1}^{\infty}[u(m)]^{-1+pqA_1}[u'(m)]^{1-p}\widetilde{a}_m^p = \frac{1}{\varepsilon}+O(1), \tag{1.52}$$

and similarly,

$$\sum_{n=1}^{\infty}[v(n)]^{-1+pqA_2}[v'(n)]^{1-q}\widetilde{b}_n^q = \frac{1}{\varepsilon}+O(1). \tag{1.53}$$

Now, suppose that there exists a positive constant M, smaller than L^*, such that (1.50) holds after replacing L^* with M. Then, combining relations (1.52) and (1.53) with inequality (1.50), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n < \frac{1}{\varepsilon}(M+o(1)). \tag{1.54}$$

On the other hand, we can also estimate the left-hand side of inequality (1.50). Namely, inserting the above defined sequences $(\widetilde{a}_m)_{m\in\mathbb{N}}$ and $(\widetilde{b}_n)_{n\in\mathbb{N}}$ in the left-hand side of (1.50), we easily obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n \\
&\quad > \int_1^{\infty}[u(x)]^{-qA_1-\frac{\varepsilon}{p}}\left[\int_1^{\infty}K(u(x),v(y))[v(y)]^{-pA_2-\frac{\varepsilon}{q}}d(v(y))\right]d(u(x)) \\
&\quad = \int_1^{\infty}[u(x)]^{-1-\varepsilon}\left[\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d(u(x)).
\end{aligned} \tag{1.55}$$

Moreover, since the kernel K is strictly decreasing in both arguments, it follows that $K(1,0) > K(1,t)$, for $t > 0$, so we have

$$\begin{aligned}
\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt &> \int_0^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt - K(1,0)\int_0^{\frac{1}{u(x)}}t^{-pA_2-\frac{\varepsilon}{q}}dt \\
&= k\left(pA_2+\frac{\varepsilon}{q}\right) - \frac{K(1,0)}{1-pA_2-\frac{\varepsilon}{q}}[u(x)]^{-1+pA_2+\frac{\varepsilon}{q}},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_1^{\infty}[u(x)]^{-1-\varepsilon}\left[\int_{\frac{1}{u(x)}}^{\infty}K(1,t)t^{-pA_2-\frac{\varepsilon}{q}}dt\right]d(u(x)) \\
&\qquad \geq \frac{1}{\varepsilon}k\left(pA_2+\frac{\varepsilon}{q}\right) - \frac{K(1,0)}{\left(1-pA_2-\frac{\varepsilon}{q}\right)\left(1-pA_2+\frac{\varepsilon}{p}\right)}.
\end{aligned} \tag{1.56}$$

In other words, relations (1.55) and (1.56) yield the lower bound for the left-hand side of inequality (1.50):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(u(m),v(n))\widetilde{a}_m\widetilde{b}_n > \frac{1}{\varepsilon}(L^*+o(1)). \tag{1.57}$$

Finally, comparing relations (1.54) and (1.57), and letting $\varepsilon \to 0^+$, it follows that $L^* \le M$, which contradicts the assumption that the constant M is smaller than L^*. This means that L^* is the best possible constant in inequality (1.50). □

We conclude this discussion with a few remarks which connect Theorems 1.14 and 1.15 with particular results presented at the beginning of this subsection.

Remark 1.12 Observe that Theorem 1.14 is a generalization of inequality (1.37) (see also [52]). Moreover, Theorem 1.15 yields the form of inequality (1.37) with the best possible constant factor. Namely, putting the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, and power functions $u(x) = Ax^{\alpha}$ and $v(y) = By^{\beta}$, A, B, α, $\beta > 0$, in (1.50), we obtain the corresponding form of inequality (1.37), with the best possible constant factor

$$\alpha^{-\frac{1}{q}}\beta^{-\frac{1}{p}}A^{-1+qA_1}B^{-1+pA_2}B(1-pA_2, 1-qA_1).$$

Remark 1.13 If $s = 1$, then parameters $A_1 = A_2 = \frac{1}{pq}$ fulfill condition (1.29). Thus, putting these parameters in (1.50) and (1.51), together with kernel $K(x,y) = (x+y)^{-1}$ and differentiable functions $u(x) = v(x) = \log(x+1)$, we obtain inequalities (1.40) and (1.41) with the best possible constants (see also [152]).

Remark 1.14 Since parameters $A_1 = A_2 = \frac{2-s}{pq}$, where $2 - \min\{p,q\} < s < 2$, fulfill condition (1.29), they can be substituted in inequalities (1.50) and (1.51). In addition, considering the kernel $K(x,y) = (x+y)^{-s}$, $s > 0$, and differentiable functions $u(x) = v(x) = x + \nu$, $0 \le \nu < 1$, inequalities (1.50) and (1.51) reduce to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n+2\nu)^s} \le S_1\left[\sum_{m=1}^{\infty}(m+\nu)^{1-s}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n+\nu)^{1-s}b_n^q\right]^{\frac{1}{q}} \tag{1.58}$$

and

$$\sum_{n=1}^{\infty}(n+\nu)^{(p-1)(s-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n+2\nu)^s}\right]^p \le S_1^p\sum_{m=1}^{\infty}(m+\nu)^{1-s}a_m^p, \tag{1.59}$$

where the constant factors $S_1 = B(\frac{1}{p} + \frac{s-1}{q}, \frac{1}{q} + \frac{s-1}{p})$ and S_1^p are the best possible. If $s = 1$, then S_1 becomes $\pi/\sin(\pi/p)$. Thus, setting $\nu = \frac{1}{2}$ and $s = 1$ in (1.58) and (1.59), we obtain the sharper version of the Hilbert double series theorem, as well as its equivalent form (see also [142]).

Remark 1.15 Some particular discrete Hilbert-type inequalities regarding homogeneous kernels are also obtained in [54]. They can be derived as the consequences of Theorems 1.14 and 1.15. For example, setting $K(x,y) = (x+y)^{-s}$, $s > 0$, $u(x) = xa^x$, $v(y) = ya^y$, $a > 1$, inequalities (1.50) and (1.51) respectively read

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(ma^m+na^n)^s} \le L^*\left[\sum_{m=1}^{\infty}(ma^m)^{-1+pqA_1}(a^m+ma^m\log a)^{1-p}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{n=1}^{\infty}(na^n)^{-1+pqA_2}(a^n+na^n\log a)^{1-p}b_n^q\right]^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty} (na^n)^{(p-1)(1-pqA_2)} (a^n + na^n \log a) \left[\sum_{m=1}^{\infty} \frac{a_m}{(ma^m + na^n)^s} \right]^p \\ \leq (L^*)^p \sum_{m=1}^{\infty} (ma^m)^{-1+pqA_1} (a^m + ma^m \log a)^{1-p} a_m^p,$$

where $L^* = B(1 - pA_2, 1 - qA_1)$.

Similarly, if $K(x,y) = (x+y)^{-s}$, $s > 0$, $u(x) = x \arctan x$, $v(y) = y \arctan y$, then inequalities (1.50) and (1.51) become

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m \arctan m + n \arctan n)^s} \leq L^* \left[\sum_{m=1}^{\infty} \omega_p(m) a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \omega_q(n) b_n^q \right]^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty} (n \arctan n)^{(p-1)(1-pqA_2)} \left(\arctan n + \frac{n}{1+n^2} \right) \\ \times \left[\sum_{m=1}^{\infty} \frac{a_m}{(m \arctan m + n \arctan n)^s} \right]^p \leq (L^*)^p \sum_{m=1}^{\infty} \omega_p(m) {a_m}^p,$$

where $L^* = B(1 - pA_2, 1 - qA_1)$ and

$$\omega_r(x) = (x \arctan x)^{1-s-r(A_2-A_1)} \left(\arctan x + \frac{x}{1+x^2} \right)^{1-r}, \ r \in \{p,q\}.$$

Of course, the above inequalities include the best possible constants. For some other examples arising from various choices of weight functions, the reader is referred to [54].

1.4.3 Some further estimates

In this subsection we study a few particular Hilbert-type inequalities involving the homogeneous kernel $K(x,y) = (x+y)^{-s}$, $s > 0$. In addition to the Hilbert inequality, the following results will be derived with a help of some additional estimates that arise from this particular setting. More precisely, we shall use Theorems 1.9 and 1.10, as well as various methods for estimating the integral of type

$$\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u) u^{-\alpha} du.$$

The first in the series of results is the following pair of inequalities.

Corollary 1.2 *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $s>0$. If $(a,b)\subseteq\mathbb{R}_+$, then the inequalities*

$$\begin{aligned}&\int_a^b\int_a^b\frac{f(x)g(y)}{(x+y)^s}dxdy\\&\le B\left(\frac{s}{2},\frac{s}{2}\right)\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]x^{-\frac{sp}{2}+p-1}f^p(x)dx\right\}^{\frac{1}{p}}\\&\qquad\times\left\{\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]y^{-\frac{sq}{2}+q-1}g^q(y)dy\right\}^{\frac{1}{q}}\end{aligned}\tag{1.60}$$

and

$$\begin{aligned}&\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{y}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{y}{b}\right)^{\frac{s}{2}}\right]^{1-p}y^{\frac{sp}{2}-1}\left[\int_a^b\frac{f(x)}{(x+y)^s}dx\right]^p dy\\&\le B^p\left(\frac{s}{2},\frac{s}{2}\right)\int_a^b\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right]x^{-\frac{sp}{2}+p-1}f^p(x)dx\end{aligned}\tag{1.61}$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$. In addition, inequalities (1.60) and (1.61) are equivalent.

Proof. Considering Theorem 1.9 with the kernel $K(x,y)=(x+y)^{-s}$ and weight functions $\varphi(x)=x^{\frac{2-s}{2q}}$, $\psi(y)=y^{\frac{2-s}{2p}}$, we have

$$\begin{aligned}&\int_a^b\int_a^b\frac{f(x)g(y)}{(x+y)^s}dxdy\\&\le\left[\int_a^b x^{-\frac{sp}{2}+p-1}\left(\int_{\frac{a}{x}}^{\frac{b}{x}}\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du\right)f^p(x)dx\right]^{\frac{1}{p}}\\&\quad\times\left[\int_a^b y^{-\frac{sq}{2}+q-1}\left(\int_{\frac{y}{b}}^{\frac{y}{a}}\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du\right)g^q(y)dy\right]^{\frac{1}{q}}.\end{aligned}$$

Now, we are going to estimate integrals in the above inequality, dependent on variables x and y. Taking into account an obvious relation

$$B\left(\frac{s}{2},\frac{s}{2}\right)=2\int_1^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du=2\alpha^{\frac{s}{2}}\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(\alpha+u)^s}du,$$

and inequality

$$\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(\alpha+u)^s}du<\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du,$$

where $\alpha>1$, we have

$$\int_\alpha^\infty\frac{u^{\frac{s}{2}-1}}{(1+u)^s}du>\frac{1}{2}\alpha^{-\frac{s}{2}}B\left(\frac{s}{2},\frac{s}{2}\right),\qquad\alpha>1.\tag{1.62}$$

Finally, considering the relation

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du = B\left(\frac{s}{2},\frac{s}{2}\right) - \int_{\frac{b}{x}}^{\infty} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du - \int_{\frac{x}{a}}^{\infty} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du,$$

and (1.62), we obtain the estimate

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{\frac{s}{2}-1}}{(1+u)^s}du < B\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}-\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}}\right],$$

which yields inequality (1.60). Equivalent form (1.61) follows in a similar way. □

Remark 1.16 Combining the well-known arithmetic-geometric mean inequality

$$\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{s}{2}}+\frac{1}{2}\left(\frac{x}{b}\right)^{\frac{s}{2}} \geq \left(\frac{a}{b}\right)^{\frac{s}{4}},$$

with inequalities (1.60) and (1.61), we have

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$
$$\leq B\left(\frac{s}{2},\frac{s}{2}\right)\left[1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right]\left[\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx\right]^{\frac{1}{p}}\left[\int_a^b y^{-\frac{sq}{2}+q-1}g(y)^q dy\right]^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{sp}{2}-1}\left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \leq \left[B\left(\frac{s}{2},\frac{s}{2}\right)\left(1-\left(\frac{a}{b}\right)^{\frac{s}{4}}\right)\right]^p\int_a^b x^{-\frac{sp}{2}+p-1}f(x)^p dx.$$

Putting $p=q=2$ in these inequalities, we obtain a pair of inequalities derived in [152]. Moreover, if $a=0$ and $b=\infty$, the above inequalities reduce to corresponding relations obtained in [11].

We finish this section with another specific Hilbert-type inequality referring to kernel $K(x,y)=(x+y)^{-s}$, $s>0$.

Corollary 1.3 *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $s>0$, *and let* A_1 *and* A_2 *be real parameters such that* $A_1\in(\frac{1-s}{q},\frac{1}{q})$ *and* $A_2\in(\frac{1-s}{p},\frac{1}{p})$. *If* $Q=k_{l_1}^{\frac{1}{p}}(pA_2)k_{l_2}^{\frac{1}{q}}(qA_1)$, $l_1=\frac{1-pA_2}{s}$, $l_2=\frac{1-qA_1}{s}$, *and*

$$k_l(\alpha)=\int_{\frac{a^l-b^l}{b(b^{l-1}-a^{l-1})}}^{\frac{a^l-b^l}{a(b^{l-1}-a^{l-1})}} \frac{u^{-\alpha}}{(1+u)^s}du,$$

then the inequalities

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy$$
$$\leq Q\left[\int_a^b x^{1-s+p(A_1-A_2)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_a^b y^{1-s+q(A_2-A_1)}g^q(y)dy\right]^{\frac{1}{q}} \quad (1.63)$$

and

$$\int_a^b y^{(p-1)(s-1)+p(A_1-A_2)} \left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le Q^p \int_a^b x^{1-s+p(A_1-A_2)} f^p(x)dx \qquad (1.64)$$

hold for all non-negative measurable functions $f,g:(a,b)\to\mathbb{R}$ and are equivalent.

Proof. We start as in the proof of Corollary 1.2, but for the estimate of the integral

$$\int_{\frac{a}{x}}^{\frac{b}{x}} \frac{u^{-\alpha}}{(1+u)^s}du,$$

we use the fact that the function $f(x)=\int_{a/x}^{b/x} u^{-\alpha}(1+u)^{-s}du$, $x\in\mathbb{R}_+$, attains its maximum value at $x=\frac{ab^l-ba^l}{a^l-b^l}$, $l=\frac{1-\alpha}{s}$. □

Remark 1.17 Setting $A_1=A_2=\frac{2-s}{pq}$, provided that $s>2-\min\{p,q\}$, inequalities (1.63) and (1.64) reduce respectively to

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy \le Q_1 \left[\int_a^b x^{1-s} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_a^b y^{1-s} g^q(y)dy\right]^{\frac{1}{q}}$$

and

$$\int_a^b y^{(s-1)(p-1)} \left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le Q_1{}^p \int_a^b x^{1-s} f^p(x)dx,$$

where

$$Q_1 = k^{\frac{1}{p}}_{\frac{q+s-2}{qs}}\left(\frac{2-s}{q}\right) k^{\frac{1}{q}}_{\frac{p+s-2}{ps}}\left(\frac{2-s}{p}\right).$$

Similarly, if $A_1=\frac{2-s}{2q}$ and $A_2=\frac{2-s}{2p}$, inequalities (1.63) and (1.64) respectively read

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^s}dxdy \le k_{\frac{1}{2}}\left(\frac{2-s}{2}\right) \left[\int_a^b x^{-\frac{sp}{2}+p-1} f(x)^p dx\right]^{\frac{1}{p}} \left[\int_a^b y^{-\frac{sq}{2}+q-1} g(y)^q dy\right]^{\frac{1}{q}}$$

and

$$\int_a^b y^{\frac{sp}{2}-1} \left[\int_a^b \frac{f(x)}{(x+y)^s}dx\right]^p dy \le k^p_{\frac{1}{2}}\left(\frac{2-s}{2}\right) \int_a^b x^{-\frac{sp}{2}+p-1} f(x)^p dx.$$

Remark 1.18 General results from this section, covering the best possible constant factors for Hilbert-type inequalities with a homogeneous kernel in integral and discrete case are established in papers [63] and [111]. Particular inequalities in subsection 1.4.3 are taken from paper [53]. For the similar problem area, the reader is referred to the following references: [13], [36], [38], [80], [85], [101], [122], [126], [132], [135], [146], [149], [155], [162], [164], [176], [181], [182], and [183].