Chapter 1

Basic notation and fundamental results

In this chapter, a brief review of some fundamental results on the topics in the sequel is given and a several basic motivating ideas are presented.

1.1 Jensen's inequality and its variants

Classical Jensen's inequality is the starting point for the variety of the results in this book. Therefore we give its discrete and integral forms in the first place, then some closely related inequalities like Jensen-Steffensen's, Jensen-Mercer's, Hölder's, Hermite-Hadamard's etc., as well as some of their numerous variants and generalizations (e.g. Jessen's inequality and its multidimensional form – McShane's inequality and, furthermore – their generalizations to the convex hulls.) Due to the close relation of Jensen's inequality to the class of convex functions, it is natural to start with the definition of convex functions. More on this topic one can find e.g. in the monographs *Convex Functions, Partial Orderings, and Statistical Applications* by J. E. Pečarić, F. Proschan and Y. L. Tong or *Convex Functions* by A. W. Roberts and D. E. Varberg.

Definition 1.1 *Let I be an interval in $\mathbb{R}$. Function $f : I \to \mathbb{R}$ is said to be a* ***convex function*** *on I if for all $x, y \in I$ and all $\lambda \in [0,1]$*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds. If (1.1) is strict for all $x, y \in I$, $x \neq y$ and for all $\lambda \in (0,1)$, then f is said to be ***strictly convex****. If the inequality in (1.1) is reversed, then f is said to be* ***concave****.*

In geometric terms, a function is convex (concave) if the part of the graph of the function between two points on that graph lies below (above) the chord which connects these two points, or, equivalently, if the epigraph of the function is a convex (concave) set.

In 1934 T. Popoviciu introduced the following generalization of the notion of convexity.

Definition 1.2 *Function $f : [a,b] \to \mathbb{R}$ is said to be* **n*-convex***, *$n \in \mathbb{N}_0$, if for every choice of mutually different points $y_0, \ldots, y_n \in [a,b]$*

$$[y_0, \ldots, y_n; f] \geq 0, \tag{1.2}$$

where $[y_0, \ldots, y_n; f]$ denotes the **n*-th divided difference*** *of the function f in $y_0, \ldots, y_n$, inductively defined with*

$$[y_i; f] = f(y_i), \quad i = 0, \ldots, n,$$

$$[y_0, \ldots, y_k; f] = \frac{[y_0, \ldots, y_{k-1}; f] - [y_1, \ldots, y_k; f]}{y_0 - y_k}, \quad k = 1, \ldots, n. \tag{1.3}$$

If (1.2) is strict, then f is said to be a ***strictly* n*-convex*** *function. If (1.2) is reversed, then f is said to be an* **n*-concave*** *function.*

Remark 1.1 According to the definition, the notion of 0-convexity corresponds to nonnegativity of the function f, 1-convexity describes the increasing function f, whereas 2-convexity corresponds to convexity in the sense of Definition 1.1. Namely, $f[x_0, x_1, x_2] \geq 0$ if and only if f is a convex function.

As we previously announced, we finally quote the Jensen inequality which can also be viewed as an alternative way of defining convex functions.

Theorem 1.1 (JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$, function $f : I \to \mathbb{R}$ be convex on I and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$. Then for any $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1.4}$$

If f is strictly convex, then (1.4) is strict, unless $x_i = c$ for all $i \in \{j : p_j > 0\}$. If f is concave, then (1.4) is reversed.

Here we also cite the accompanied reversed inequality for convex functions.

Theorem 1.2 (REVERSED JENSEN'S INEQUALITY) *Let I be an interval in $\mathbb{R}$, function $f: I \to \mathbb{R}$ be convex on I and let $p = (p_1, \dots, p_n)$ be a real n-tuple such that $p_1 > 0$, $p_i \leq 0$, $i = 2, \dots, n$, $P_n = \sum_{i=1}^n p_i > 0$. Then for $x_i \in I$ $(i = 1, \dots, n)$, such that $\frac{1}{P_n} \sum_{i=1}^n p_i x_i \in I$ the following inequality holds:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{1.5}$$

It is worth mentioning that when Danish mathematician J. L. W. Jensen established the inequality (1.4) in 1905 (see [91]), he originally did it for the class of midconvex (Jensen-convex) functions, that is for the class of functions for which

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}. \tag{1.6}$$

Since comparison of means is at the core of the notion of convexity, let us firstly recall some of the basic related definitions, with an accent on means that we are going to use extensively in the following chapters. For more details on this subject, the reader may be referred e.g. to [165].

Definition 1.3 *Let $M: I \times I \to I$ be a continuous function, where I is an interval in $\mathbb{R}$. If M satisfies the condition*

$$\inf\{s,t\} \leq M(s,t) \leq \sup\{s,t\}, \quad \textit{for all} \quad s,t \in I,$$

*then we say that M is a **mean** on the interval I.*

The weight combinations $M(\mathbf{x}, \mathbf{p})$, where $\mathbf{x}$ and $\mathbf{p}$ are positive real n-tuples, $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, such that $\sum_{i=1}^n p_i = 1$, can be defined in the same manner, with the condition

$$\inf\{x_1, \dots, x_n\} \leq M(\mathbf{x}, \mathbf{p}) \leq \sup\{x_1, \dots, x_n\}, \quad \text{for all} \quad x_1, \dots, x_n \in I.$$

Let $n \in \mathbb{N}$ and let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ be positive real n-tuples such that $\sum_{i=1}^n p_i = 1$. A **quasi-arithmetic mean** associated to a strictly monotonic continuous function $\varphi: I \to \mathbb{R}$ is defined by

$$M_\varphi(\mathbf{x}; \mathbf{p}) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i)\right).$$

For $n \in \mathbb{N}$ and for positive real n-tuples $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$, such that $P_n = \sum_{i=1}^n p_i > 0$, a **weight power mean** of order r of $\mathbf{x}$ is defined by

$$M_r(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \in \mathbb{R},\ r \neq 0; \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0; \\ \min\{x_1, \dots, x_n\}, & r \to -\infty; \\ \max\{x_1, \dots, x_n\}, & r \to \infty. \end{cases} \tag{1.7}$$

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Note that for $\varphi(x) = x^r$ the weight power mean can be obtained as a special case of the quasi-arithmetic mean. The following means are thus the special cases of the weight power mean:

(i) $M_1(\mathbf{x},\mathbf{p}) = A_n = \sum_{i=1}^{n} p_i x_i$ ... arithmetic mean,

(ii) $M_0(\mathbf{x},\mathbf{p}) = G_n = \prod_{i=1}^{n} x_i^{p_i}$ ... geometric mean,

(iii) $M_{-1}(\mathbf{x},\mathbf{p}) = H_n = \dfrac{1}{\sum_{i=1}^{n} \frac{p_i}{x_i}}$ ... harmonic mean.

Stolarsky's means are another class of means which are of interest for us in some of the following chapters. These are defined by

$$S_p(s,t) = \left[\frac{s^p - t^p}{ps - pt}\right]^{\frac{1}{p-1}}, \quad p \neq 0,1.$$

The limiting cases $p = 0$ and $p = 1$ provide the definitions of the **logarithmic** and the **identric** means, respectively:

$$\begin{aligned} S_0(s,t) &= \lim_{p\to 0} S_p(s,t) = \frac{s-t}{\log s - \log t} = L(s,t), \\ S_1(s,t) &= \lim_{p\to 1} S_p(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}} = I(s,t). \end{aligned}$$

After this short digression, we go back to analyzing the Jensen inequality. Notice that $M_1(\mathbf{x},\mathbf{p}) = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ represents a form of the weight arithmetic mean of $x_1,\ldots,x_n$. Hence Jensen's inequality (1.4) assumes the following form:

$$f(M_1(x,\mathbf{p})) \leq M_1(f(x),\mathbf{p}).$$

There are many integral variants of the Jensen inequality. The proof of the following theorem can be found e.g. in ([177, p. 45]).

Theorem 1.3 (INTEGRAL JENSEN'S INEQUALITY) *Let $(\Omega,\mathscr{A},\mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\varphi : \Omega \to \mathbb{R}$ be a μ-integrable function. Let $f : I \to \mathbb{R}$ be a convex function such that $Im\,\varphi \subseteq I$ and $f \circ \varphi$ is a μ-integrable function. Then*

$$f\left(\frac{1}{\mu(\Omega)}\int_\Omega \varphi(x)d\mu(x)\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega f(\varphi(x))d\mu(x), \tag{1.8}$$

or: $f(M_1(\varphi;\mu)) \leq M_1(f\circ\varphi;\mu)$, where $M_1(\varphi;\mu) = \dfrac{1}{\mu(\Omega)}\displaystyle\int_\Omega \varphi(x)d\mu(x)$, $M_1(\varphi;\mu) \in I$. If f is a strictly convex function, then (1.8) *becomes equality if and only if φ is a constant μ – almost everywhere on Ω. If f is a concave function, then* (1.8) *is reversed.*

Now the discrete Jensen inequality (1.4) is obtained by means of the discrete measure μ on $\Omega = \{1, \ldots, n\}$, with $\mu(\{i\}) = p_i$, and $\varphi(i) = x_i$.

Another integral variant of Jensen's inequality is based on the notion of the Riemann-Stieltjes integral, for which a brief outline is given here. One can find more information on the Riemann-Stieltjes integral in ([195]).

Let $[a,b] \subset \mathbb{R}$ and let $f, \varphi : [a,b] \to \mathbb{R}$ be bounded functions. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$, such that $t_0 < t_1 < \cdots < t_{n-1} < t_n$ Stieltjes' integral sum

$$\sigma(f, \varphi; D, \gamma_1, \ldots, \gamma_n) = \sum_{i=1}^{n} f(\gamma_i)(\varphi(t_i) - \varphi(t_{i-1}))$$

is assigned, where γ_i are from $[t_{i-1}, t_i]$, $i \in \{1, 2, \ldots, n\}$. These sums will be denoted with $\sigma(f, \varphi; D)$ in the sequel.

Definition 1.4 *Let $f, \varphi : [a,b] \to \mathbb{R}$ be bounded functions. Function f is said to be* ***Riemann-Stieltjes integrable*** *regarding function φ if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exits a decomposition D_0 of $[a,b]$, such that for every decomposition $D \supseteq D_0$ of $[a,b]$ and for every sum $\sigma(f, \varphi; D)$*

$$|\sigma(f, \varphi; D) - I| < \varepsilon$$

holds. The unique number I is the ***Riemann-Stieltjes integral*** *of the function f regarding the function φ and is denoted with*

$$\int_a^b f(t)\, d\varphi(t). \tag{1.9}$$

The Riemann-Stieltjes integral is a generalization of the Riemann integral and coincides with it when φ is an identity.

The notion of the Riemann-Stieltjes integral is narrowly related to the class of the functions of bounded variation.

Let $\varphi : [a,b] \to \mathbb{R}$ be a real function. To each decomposition $D = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ such that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{1.10}$$

belongs the sum

$$V(\varphi; D) = \sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})|,$$

which is said to be a **variation of the function** φ regarding decomposition D.

Definition 1.5 *Function $\varphi : [a,b] \to \mathbb{R}$ is said to be a* ***function of bounded variation*** *if the set $\{V(\varphi; D) : D \in \mathscr{D}\}$ is bounded, where $\mathscr{D}$ is a family of all decompositions of the interval* (1.10)*. Number*

$$V(\varphi) = \sup\{V(\varphi; D) : D \in \mathscr{D}\}$$

is called a ***total variation*** *of the function φ.*

Theorem 1.4 *The following assertions hold:*

a) *Every monotonic function $f : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$ and $V_a^b(f) = |f(b) - f(a)|$;*

b) *Every function of bounded variation is a bounded function;*

c) *If f and g are functions of bounded variation on $[a,b]$, then $f+g$ is a function of bounded variation on $[a,b]$.*

Theorem 1.5 *Let f be a function of bounded variation on $[a,b]$. Then*

a) *f has at most countably many of step discontinuities on $[a,b]$;*

b) *f can be presented as $f = s_f + g$, where step-function s_f and continuous function g are both functions of bounded variation on $[a,b]$.*

Theorem 1.6 *Let $f : [a,b] \to \mathbb{R}$ be a continuous function and $\varphi : [a,b] \to \mathbb{R}$ be a function of bounded variation. Then there exists the Riemann-Stieltjes integral* (1.9) *and*

$$\left| \int_a^b f(t)\, d\varphi(t) \right| \leq V(\varphi) \cdot \max_{t \in [a,b]} |f(t)|.$$

Regarding the Riemann-Stieltjes integral, we now induce yet another integral form of Jensen's inequality, dealt with in one of the following chapters (for more details on this topic, the reader is referred to [177, p. 58]). It reads as follows:

$$f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta g(t) \mathrm{d}\lambda(t) \right) \leq \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta f(g(t)) \mathrm{d}\lambda(t), \tag{1.11}$$

where $g : [\alpha,\beta] \to (a,b)$ is a continuous function, $-\infty < \alpha < \beta < \infty$, $-\infty \leq a < b \leq \infty$, $f : (a,b) \to \mathbb{R}$ is a convex function and $\lambda : [\alpha,\beta] \to \mathbb{R}$ is an increasing function, such that $\lambda(\beta) \neq \lambda(\alpha)$.

In 1919 J. F. Steffensen proved that inequality (1.4) held when the condition on non-negativity of the n-tuple $\mathbf{p}$ was relaxed, but with simultaneously restricted choice on $\mathbf{x}$. In a more general form, Steffensen's theorem reads as follows.

Theorem 1.7 (JENSEN-STEFFENSEN'S INEQUALITY) *If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function, $\mathbf{x} \in I^n$ is a monotonic n-tuple and $\mathbf{p}$ is a real n-tuple such that*

$$P_n > 0 \quad \textit{and} \quad 0 \leq P_k \leq P_n, \quad 1 \leq k \leq n-1,$$

where $P_k = \sum_{i=1}^k p_i$, $k = 1, \ldots, n$, then inequality (1.4) *holds.*

Recently, J. E. Pečarić provided yet another proof of Theorem 1.7 and one can find it in [177, p. 57]. Integral variants of the previous theorem will be discussed in one of the following chapters.

In 2003 A. McD. Mercer proved yet another variant of Jensen's inequality (see [134]). In a slightly generalized form his theorem is stated as below.

Theorem 1.8 (JENSEN-MERCER'S INEQUALITY) *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\mathbf{x} = (x_1,\dots,x_n) \in [a,b]^n$. Suppose $\mathbf{p} = (p_1,\dots,p_n)$ is a nonnegative real n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$. If $f : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality holds:*

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \le f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1.12}$$

What is of an additional interest here is that in 2005 S. Abramovich *et.al.* in [2] proved another variant of Jensen-Steffensen's inequality, which included Mercer's original result as its special case. Jensen-Mercer's inequality was proved under the Steffensen's conditions as in Theorem 1.7.

Theorem 1.9 (see [2]) *Let $[a,b]$ be an interval in $\mathbb{R}$ and let $\mathbf{x} \in [a,b]^n$ be a monotonic n-tuple. Suppose $\mathbf{p}$ is a real n-tuple, such that*

$$P_n > 0 \quad \text{and} \quad 0 \le P_k \le P_n, \quad 1 \le k \le n-1, \tag{1.13}$$

where $P_k = \sum_{i=1}^k p_i$, $k = 1,2,\dots,n$. If $f : [a,b] \to \mathbb{R}$ is a convex function, then inequality (1.12) *holds.*

An integral variant of the previous theorem will be discussed in one of the following chapters.

Strongly related to Jensen's inequality is the converse Jensen inequality. Although there are more variants of its converses, some of which are going to be explored in one of the following chapters, here we single out the Lah-Ribarič inequality as one of the most significant ones (see [125] or, for example, [151, p. 9]).

Theorem 1.10 (LAH-RIBARIČ) *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, $x_i \in [a,b]$, $p_i \ge 0$, $i = 1,\dots,n$ and $\sum\limits_{i=1}^n p_i = 1$. Then the following inequality holds:*

$$\sum_{i=1}^n p_i f(x_i) \le \frac{b-\sum_{i=1}^n p_i x_i}{b-a} f(a) + \frac{\sum_{i=1}^n p_i x_i - a}{b-a} f(b). \tag{1.14}$$

If f is strictly convex, then (1.14) *is strict unless $x_i \in \{a,b\}$, for all $i \in \{j : p_j > 0\}$.*

In 1931 Jensen's inequality (1.4) was investigated by B. C. Jessen, who generalized it by means of the positive linear functional acting on a space of real functions.

Let E be a nonempty set and L a linear class of functions $f : E \to \mathbb{R}$ which possesses the following properties:

L1: If $f,g \in L$, then $\alpha f + \beta g \in L$, for all α, $\beta \in \mathbb{R}$;

L2: $1 \in L$, that is, if $f(x) = 1$, $x \in E$, then $f \in L$.

We consider positive linear functionals $A : L \to \mathbb{R}$, or in other words we assume:

A1: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$, for $f,g \in L$ and $\alpha,\beta \in \mathbb{R}$;

A2: If $f(x) \geq 0$ for all $x \in E$, then $A(f) \geq 0$.

If additionally the condition

A3: $A(1) = 1$

is satisfied, we say that A is a normalized positive linear functional or that $A(f)$ is a linear mean on L.

In the described environment we cite the Jessen's result.

Theorem 1.11 (JESSEN'S INEQUALITY) *Let E be a nonempty set and let L be a linear class of functions $f : E \to \mathbb{R}$ which possesses the properties L1 and L2. Suppose $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is a continuous and convex function. If $A : L \to \mathbb{R}$ is a normalized positive linear functional, then for all $f \in L$, such that $\Phi(f) \in L$ we have $A(f) \in I$ and the following inequality holds:*

$$\Phi(A(f)) \leq A(\Phi(f)). \tag{1.15}$$

In 1937 E. J. McShane gave an important generalization of Jessen's inequality, in his paper [133]. He observed Φ in (1.15) as a function of several variables. Namely, vector function $\mathbf{f} : E \to \mathbb{R}^n$ was defined with $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))$, where $f_i \in L$, $i = 1, \dots, n$. Such multidimensional generalization of (1.15) is described in the following theorem.

Theorem 1.12 (MCSHANE'S INEQUALITY) *Let E be a nonempty set and let L be a linear class of real functions defined on E, which possesses the properties L1 and L2. Let $K \subseteq \mathbb{R}^n$ be a closed convex set and let $\Phi : K \to \mathbb{R}$ be a continuous convex function. If $A : L \to \mathbb{R}$ is a normalized positive linear functional, then for all functions $\mathbf{f} = (f_1, \dots, f_n) \in L^n$, such that $\Phi(\mathbf{f}) \in L$ we have $A(\mathbf{f}) \in K$ and the following inequality holds:*

$$\Phi(A(\mathbf{f})) \leq A(\Phi(\mathbf{f})). \tag{1.16}$$

In the previous theorem, acting of the functional A to the vector function $\mathbf{f} = (f_1, \dots, f_n)$ is defined with $A(\mathbf{f}) = (A(f_1), \dots, A(f_n))$.

One can find the proofs of the theorems 1.11 and 1.12 in [177, from p. 47].

When dealing with the positive normalized linear functionals, we need to mention that in 1985 J. Pečarić and P. R. Beesack presented a corresponding generalization of the Theorem 1.10. Namely, they proved that for a convex function f defined on an interval $I = [m, M] \subset \mathbb{R}$, $(-\infty < m < M < \infty)$ and for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$ the following inequality holds:

$$A(f(g)) \leq \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \tag{1.17}$$

As for the generalized forms of the Jensen-type inequalities in this setting, let us mention here the generalization of the Jensen-Mercer inequality (1.12) which involves positive normalized linear functionals and is called the Jessen-Mercer inequality.

Theorem 1.13 (JESSEN-MERCER'S INEQUALITY) *Let L satisfy L1, L2 on a nonempty set E, and let A be a positive normalized linear functional. If φ is a continuous convex*

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function on $[m,M]$*, then for all* $f \in L$ *such that* $\varphi(f), \varphi(m+M-f) \in L$ *(so that* $m \leq f(t) \leq M$ *for all* $t \in E$*), we have*

$$\varphi(m+M-A(f)) \leq \varphi(m) + \varphi(M) - A(\varphi(f)). \tag{1.18}$$

If the function φ *is concave, then* (1.18) *is reversed.*

In some of the following chapters we deal with the generalizations of Jensen's and related inequalities on convex hulls in $\mathbb{R}^k$ and, as a special case, on k-simplices in $\mathbb{R}^k$. For that purpose we define the mentioned notions.

The **convex hull** of the vectors $x_1, \ldots, x_n \in \mathbb{R}^k$ is the set

$$K = co(\{x_1, \ldots, x_n\}) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \middle| \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Barycentric coordinates over K are continuous real functions $\lambda_1, \ldots, \lambda_n$ on K with the following properties:

- $$\lambda_i(x) \geq 0, \quad i = 1, \ldots, n,$$
  $$\sum_{i=1}^{n} \lambda_i(x) = 1,$$
  $$x = \sum_{i=1}^{n} \lambda_i(x) x_i. \tag{1.19}$$

The **k-simplex** $S = [v_1, \ldots, v_{k+1}]$ is a convex hull of its vertices $v_1, \ldots, v_{k+1} \in \mathbb{R}^k$, where $v_2 - v_1, \ldots, v_{k+1} - v_1 \in \mathbb{R}^k$ are linearly independent.

As an illustrative example serves a generalization of the result (1.17) that was obtained in [88], where for $x_1, \ldots, x_n \in \mathbb{R}^k$, $K = co(\{x_1, \ldots, x_n\})$, as well as for a convex function f on K, barycentric coordinates $\lambda_1, \ldots, \lambda_n$ over K and for all $g \in L^k$ such that $g(E) \subset K$ and $f(g), \lambda_i(g) \in L$, $i = 1, \ldots, n$, the inequality

$$A(f(g)) \leq \sum_{i=1}^{n} A(\lambda_i(g)) f(x_i) \tag{1.20}$$

holds.

1.1.1 n-exponentially and exponentially convex functions

Notions of n-exponentially and exponentially convex functions are going to be explored in some of the following chapters. For that purpose we define them here and provide some of their characterizations.

Definition 1.6 *A function* $\psi : I \to \mathbb{R}$ *is* **n-*exponentially convex in the Jensen sense*** *on* I *if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ *and* $x_i \in I$, $i = 1,\dots,n$.

A function $\psi\colon I \to \mathbb{R}$ *is* ***n-exponentially convex*** *if it is n-exponentially convex in the Jensen sense and is continuous on I.*

Remark 1.2 It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \le n$.

The following proposition follows by the definition of positive semi-definite matrices and by utilizing some basic linear algebra.

Proposition 1.1 *If* ψ *is an n-exponentially convex function in the Jensen sense, then the matrix* $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ *is a positive semi-definite matrix for all* $k \in \mathbb{N}$, $k \le n$. *Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \ge 0$, *for all* $k \in \mathbb{N}$, $k \le n$.

Definition 1.7 *A function* $\psi\colon I \to \mathbb{R}$ *is* ***exponentially convex in the Jensen sense*** *on I if it is n-exponentially convex in the Jensen sense for all* $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ *is* ***exponentially convex*** *if it is exponentially convex in the Jensen sense and is continuous on I.*

Definition 1.8 *A positive function* ψ *is said to be* ***logarithmically convex*** *(or* ***log-convex****) on an interval* $I \subseteq \mathbb{R}$ *if* $\log\psi$ *is a convex function on I, or equivalently, if*

$$\psi(\lambda x + (1-\lambda)y) \le \psi^{\lambda}(x)\psi^{1-\lambda}(y)$$

holds for all $x,y \in I$ *and* $\lambda \in [0,1]$.

A positive function ψ *is* ***log-convex in the Jensen sense*** *if*

$$\psi^2\left(\frac{x+y}{2}\right) \le \psi(x)\psi(y)$$

holds for all $x,y \in I$, *i.e., if* $\log\psi$ *is convex in the Jensen sense.*

Remark 1.3 It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \ge 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x,y \in I$. It follows that a function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see for example [177]).

Proposition 1.2 *If Ψ is a convex function on an interval I and if $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$, then the following inequality is valid:*

$$\frac{\Psi(x_2)-\Psi(x_1)}{x_2-x_1} \le \frac{\Psi(y_2)-\Psi(y_1)}{y_2-y_1}. \tag{1.21}$$

If the function Ψ is concave, the inequality reverses.

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

Remark 1.4 Definition 1.2 provided the notion of the second order divided difference, needed in the sequel. The value $[y_0,y_1,y_2;f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$, we get

$$\lim_{y_1\to y_0}[y_0,y_1,y_2;f] = [y_0,y_0,y_2;f] = \frac{f(y_2)-f(y_0)-f'(y_0)(y_2-y_0)}{(y_2-y_0)^2}, \quad y_2 \ne y_0,$$

provided f' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1,2$ we get

$$\lim_{y_2\to y_0}\lim_{y_1\to y_0}[y_0,y_1,y_2;f] = [y_0,y_0,y_0;f] = \frac{f''(y_0)}{2},$$

provided that f'' exists.

We will use an idea from [90] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions, applying some functionals to a given family with the same property.

1.2 Some classical inequalities

In this section we give an outline of some important classical inequalities to which we will often refer throughout the following chapters. Namely, refinements and converses of arithmetic-geometric, geometric-harmonic inequalities, as well as Young's, Hölder's, Minkowski's, Hilbert's and some other classical inequalities are going to be presented throughout this monograph. The reader can find more details on these topics, as well as the results with the corresponding proofs e.g. in [151], [165] or [177].

Theorem 1.14 (WEIGHT ARITHMETIC-GEOMETRIC MEAN INEQUALITY) *Let $n \in \mathbb{N}$, $n \ge 2$, $x_1,\dots,x_n > 0$, $\alpha_1,\dots,\alpha_n \in (0,1)$ such that $\sum_{i=1}^n \alpha_i = 1$. Then the inequality*

$$\sum_{i=1}^n \alpha_i x_i \ge \prod_{i=1}^n x_i^{\alpha_i} \tag{1.22}$$

holds. Equality holds for $x_1 = \cdots = x_n$.

Corollary 1.1 (WEIGHT GEOMETRIC-HARMONIC MEAN INEQUALITY) *Let* $n \in \mathbb{N}$, $n \geq 2$, $x_1, \ldots, x_n > 0$, $\alpha_1, \ldots, \alpha_n \in (0,1)$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$. *Then*

$$\prod_{i=1}^{n} x_i^{\alpha_i} \geq \frac{1}{\sum_{i=1}^{n} \frac{\alpha_i}{x_i}} \tag{1.23}$$

holds. Equality holds for $x_1 = \cdots = x_n$.

Remark 1.5 From (1.22) and (1.23) and for $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$ we get the classical arithmetic-geometric-harmonic mean inequality:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x^i\right)^{\frac{1}{n}} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}, \tag{1.24}$$

with corresponding equalities obtained for $x_1 = \cdots = x_n$.

Family of so called Heinz means, denoted with H_ν interpolates arithmetic and geometric mean of nonnegative real numbers a and b and is defined by

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}, \qquad \nu \in [0,1]. \tag{1.25}$$

Obviously,

$$\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}. \tag{1.26}$$

Following inequality is closely related to the arithmetic-geometric mean inequality.

Theorem 1.15 (YOUNG'S INEQUALITY) *Let* $f : [0,\infty] \to [0,\infty]$ *be an increasing continuous function such that* $f(0) = 0$ *and* $\lim_{x\to\infty} f(x) = \infty$. *Then for all* $a, b \geq 0$

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx, \tag{1.27}$$

holds. Equality holds if and only if $b = f(a)$.

Remark 1.6 If the function f in Theorem 1.15 is defined with $f(x) = x^{p-1}$, $p > 1$, we get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.28}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and the connection with the arithmetic-geometric mean inequality becomes obvious.

Young's inequality is a starting point for Hölder's inequality.

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Theorem 1.16 (DISCRETE HÖLDER'S INEQUALITY) *Let a_i, b_i, $i = 1,\dots,n$ be complex numbers and for $p > 1$ let q be defined by $\frac{1}{p}+\frac{1}{q}=1$. Then for all $n \in \mathbb{N}$ the following inequality holds:*

$$\sum_{i=1}^{n}|a_i b_i| \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|b_i|^q\right)^{\frac{1}{q}}. \tag{1.29}$$

Equality in (1.29) holds if and only if the n-tuples $a_1,\dots,a_n$ and $b_1,\dots,b_n$ are proportional.

Remark 1.7 For $p = 2$ inequality (1.29) is the well known Cauchy-Schwarz inequality.

Hölder's inequality can be observed in a more general environment, involving the positive linear functionals acting on the space of real functions. For that purpose we refer to the notation induced in the previous Section 1.1 and cite the following result.

Theorem 1.17 *Let E be a nonempty set and L be a linear class of real functions defined on E, which satisfies properties L1 and L2. Suppose $p_i > 1$, $i = 1,\dots,n$ are such that $\sum_{i=1}^{n}\frac{1}{p_i}=1$. Let $f_i \in L$, $i = 1,\dots,n$ be nonnegative functions, such that $\prod_{i=1}^{n} f_i^{\frac{1}{p_i}} \in L$ is a nonnegative function. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$A\left(\prod_{i=1}^{n} f_i^{\frac{1}{p_i}}\right) \leq \prod_{i=1}^{n} A^{\frac{1}{p_i}}(f_i). \tag{1.30}$$

The Minkowski inequality can also be observed in the discrete and in a more general setting.

Theorem 1.18 (DISCRETE MINKOWSKI'S INEQUALITY) *Let a_i, b_i, $i = 1,\dots,n$ be complex numbers and let $p \geq 1$. Then for all $n \in \mathbb{N}$ the following inequality holds:*

$$\left(\sum_{i=1}^{n}|a_i+b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|b_i|^p\right)^{\frac{1}{p}}. \tag{1.31}$$

Theorem 1.19 *Let E be a nonempty set and L be a linear class of real functions defined on E, which satisfies properties L1 and L2. Suppose $p \geq 1$. Let $f_i \in L$, $i = 1,\dots,n$ be nonnegative functions, such that f_i^p, $(\sum_{i=1}^{n} f_i)^p \in L$. If $A : L \to \mathbb{R}$ is a positive linear functional, then the following inequality holds:*

$$A^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n} f_i\right)^p\right] \leq \sum_{i=1}^{n} A^{\frac{1}{p}}(f_i^p). \tag{1.32}$$

In the early years of the last century two fundamental inequalities were proved. The first one was discrete.

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Theorem 1.20 *Let $(a_m)_{m\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be nonnegative real sequences such that $\sum_{m=1}^{\infty} a_m^p < \infty$ and $\sum_{n=1}^{\infty} b_n^p < \infty$. Suppose that for $p > 1$ q is defined by $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.33}$$

unless $(a_m)_{m\in\mathbb{N}}$ or $(b_n)_{n\in\mathbb{N}}$ is a null-sequence.

The second inequality was obtained in the integral form.

Theorem 1.21 *Let f and g be nonnegative integrable functions such that $\int_0^\infty f^p(x)dx < \infty$ and $\int_0^\infty g^q(y)dy < \infty$. Suppose that for $p > 1$ is q defined by $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.34}$$

unless f or g is a null-function.

The bilinear form $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}$ from (1.33) was in the first place investigated and estimated by D. Hilbert in the nineteenth century, and thus both inequalities were named after him as Hilbert's inequalities. Their significance became obvious later in the 20th century, for its many generalizations, improvements and various proofs have been given by numerous famous mathematicians, for example, L. Fejér, G. H. Hardy, J. Littlewood, G. Polya, I. Schur and many others. The detailed approach to this subject was given in the monograph [83]. Recent results on Hilbert's inequality, including the following one that unifies its discrete and the integral case are presented in the monograph [122].

Theorem 1.22 *Let $\Omega \subseteq (0,\infty)$. Suppose p and q are conjugate parameters, i.e. $\frac{1}{p}+\frac{1}{q}=1$ such that $p>1$ and let $K\colon \Omega\times\Omega\to\mathbb{R}$, $\varphi\colon\Omega\to\mathbb{R}$ and $\psi\colon\Omega\to\mathbb{R}$ are nonnegative measurable functions. Let $F,G\colon\Omega\to\mathbb{R}$ be defined by*

$$F(x) = \int_\Omega \frac{K(x,y)}{\psi^p(y)}d\mu_2(y) \quad \text{and} \quad G(y) = \int_\Omega \frac{K(x,y)}{\varphi^q(x)}d\mu_1(x), \tag{1.35}$$

where μ_1 and μ_2 are σ-finite measures. Then for any choice of nonnegative measurable functions $f,g\colon\Omega\to\mathbb{R}$ the following inequality holds:

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)f(x)g(y)d\mu_1(x)d\mu_2(y)\\ &\leq \left[\int_\Omega \varphi^p(x)F(x)f^p(x)d\mu_1(x)\right]^{\frac{1}{p}}\left[\int_\Omega \psi^q(y)G(y)g^q(y)d\mu_2(y)\right]^{\frac{1}{q}}.\end{aligned} \tag{1.36}$$

Many important inequalities are established for the class of convex functions, but one of the most famous is the Hermite-Hadamard inequality. This double inequality, which was first discovered by C. Hermite in 1881, is stated as follows (see for example [177, p. 137]).

Theorem 1.23 (HERMITE-HADAMARD) *Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.37}$$

This result was later incorrectly attributed to J. S. Hadamard who apparently was not aware of Hermite's discovery and today, when relating to (1.37), we use both names. It is interesting to mention that the term *convex* also stems from a result obtained by Hermite in 1881.

Note that the first inequality in (1.37) is stronger than the second one.

The following inequality will be referred to as Hammer-Bullen's in the sequel. Its geometric proof was given in [79] and the analytic one in [46] (see also [177, p. 140]).

Theorem 1.24 (HAMMER-BULLEN) *Let f be a convex function on $[a,b] \subset \mathbb{R}$, where $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{1.38}$$

For the sake of further considerations throughout this monograph, we conclude this section with recalling two famous theorems: the Lagrange and the Cauchy mean value theorems, where the first one is a special case of the latter.

Theorem 1.25 (LAGRANGE MEAN VALUE THEOREM) *If a function f is continuous on the closed interval $[a,b]$, where $a < b$ and differentiable on the open interval (a,b), then there exists point c in (a,b) such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Theorem 1.26 (CAUCHY MEAN VALUE THEOREM) *If f and g are continuous functions on the closed interval $[a,b]$, where $a < b$, if $g(a) \neq g(b)$, and both functions are differentiable on the open interval (a,b), then there exists at least one point c in (a,b), $a < c < b$, such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

1.3 C*-algebras. Operators on a Hilbert space

Dealing with the Jensen-type functionals whose real arguments are substituted with the bounded self-adjoint operators acting on a Hilbert space, the ones that we are also going to investigate in the sequel, requires additional overview of some basic notions and results. These concern C*-algebras and the theory on operators on a Hilbert space. For a more detailed analysis, the reader is referred to e.g. [15] and [74].

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Recall that a linear space $\mathscr{A}$ over the field $\mathbf{F}$, together with the multiplication $(x,y) \mapsto xy$, $x,y \in \mathscr{A}$, constitutes an *algebra* if the mapping $(x,y) \mapsto xy$ possesses the following properties:

$$(xy)z = x(yz), \quad x(y+z) = xy + xz, \quad (x+y)z = xz + yz,$$
$$(\alpha x)y = \alpha(xy) = x(\alpha y),$$

for all $\alpha \in \mathbf{F}$ and for all $x,y \in \mathscr{A}$. If $\mathbf{F} = \mathbb{R}$, algebra is called a real algebra and if $\mathbf{F} = \mathbb{C}$, we call it a complex algebra. If there is an element 1 in $\mathscr{A}$ such that $x \cdot 1 = 1 \cdot x = x$, for all $x \in \mathscr{A}$, then we say that $\mathscr{A}$ is an algebra with the unit 1.

Mapping $x \mapsto \|x\|$ defined on an algebra $\mathscr{A}$ and with the values in $\mathbb{R}$ is a *norm* on $\mathscr{A}$ if the following conditions are satisfied: $x \mapsto \|x\|$ is a norm on the linear space of $\mathscr{A}$, $\|xy\| \leq \|x\|\|y\|$, $x,y \in \mathscr{A}$, and, if $\mathscr{A}$ has a unit 1, then $\|1\| = 1$. Ordered pair $(\mathscr{A}, \|\cdot\|)$ is called a *normed algebra*. Normed algebra is called a *Banach algebra* if the normed space of $\mathscr{A}$ is a Banach space, that is a complete normed vector space.

Involution on algebra $\mathscr{A}$ is a conjugate linear mapping $x \mapsto x^*$ on $\mathscr{A}$, such that $x^{**} = x$ and $(xy)^* = y^*x^*$, for all $x,y \in \mathscr{A}$. Ordered pair $(\mathscr{A}, *)$ is called an *involutive algebra* or a **-algebra*. Now, a *-algebra $\mathscr{A}$ equipped with the complete submultiplicative norm such that $\|x^*\| = \|x\|$, for all $x \in \mathscr{A}$ constitutes a *Banach *-algebra*.

Finally, Banach *-algebra is a *C*-algebra* if for every $x \in \mathscr{A}$, C*-identity $\|x^*x\| = \|x\|^2$ holds. We say that C*-algebra is unital if it contains the unit 1. The element $a \in \mathscr{A}$ is called: self-adjoint if $a = a^*$, normal if $a^*a = aa^*$ and is called unitary (in the unital algebra) if $a^*a = aa^* = 1$. A standard example of the unital C*-algebra is the scalar field $\mathbb{C}$, where the involution is represented as the complex conjugation.

The environment that is of an interest for us in the sequel is the one of the bounded linear operators on a Hilbert space, which we therefore specify here.

A Hilbert space H is a (complex) vector space H that is complete regarding the metric $d(x,y) = \|x-y\|$ defined by the norm which is induced by an inner product $\|x\| := \langle x,x\rangle^{\frac{1}{2}}$. In other words, Hilbert space is a complete unitary space.

A linear operator A on a Hilbert space H is bounded if

$$\|A\| := \sup\{\|Ax\| : \|x\| \leq 1, x \in H\} < \infty.$$

We say that $\|A\|$ is an operator norm of A. The sum and the composition of the bounded linear operators is again a bounded linear operator. The mapping $(x,y) \mapsto \langle Ax,y\rangle$ is linear and continuous and according to the Riesz representation theorem (see e.g. [16]), it follows that

$$\langle x, A^*y\rangle = \langle Ax,y\rangle, \quad \text{for} \quad A^*y \in H.$$

Thus another bounded linear operator A^* on H is defined and $A^{**} = A$. Bounded (hence continuous) linear operators on H together with an operator norm and the corresponding involution constitute a C*-algebra denoted with $\mathscr{B}(H)$. *Spectrum* of an operator $A \in \mathscr{B}(H)$ is defined with

$$\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda 1_H \, \text{not invertible in} \, \mathscr{B}(H)\},$$

where 1_H is a unit operator on H. This set is non-empty and compact for operators in $\mathscr{B}(H)$. A bounded linear operator A on a Hilbert space H is self-adjoint if $A = A^*$. The following characterization holds: $A \in \mathscr{B}(H)$ is self-adjoint if and only if $\langle Ax, x\rangle \in \mathbb{R}$, for all $x \in H$.

Bounded self-adjoint operators constitute the subspace of the C*-algebra of all bounded linear operators and is denoted with $\mathscr{B}_h(H)$. We induce the partial ordering in $\mathscr{B}_h(H)$.

Definition 1.9 *Operator $A \in \mathscr{B}_h(H)$ is* ***positive semidefinite*** *or* ***positive*** *($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in H$. Positive semidefinite operator $A \in \mathscr{B}_h(H)$ is* ***positive definite*** *or* ***strictly positive*** *($A > 0$) if there is a real number $m > 0$ such that $\langle Ax, x\rangle \geq m\langle x, x\rangle$, $x \in H$, that is $A \geq m1_H$. For operators $A, B \in \mathscr{B}_h(H)$ is $B \geq A$ or $A \leq B$ if $B - A \geq 0$, that is if $\langle Bx, x\rangle \geq \langle Ax, x\rangle$, for all $x \in H$. Such ordering is called an* ***operator ordering****. In particular, for scalars m and M is $m1_H \leq A \leq M1_H$ if $m \leq \langle Ax, x\rangle \leq M$, for every unit vector $x \in H$.*

If for a self-adjoint operator A is $\sigma(A) \subseteq [m, M]$, then $m1_H \leq A \leq M1_H$.

The set of all positive operators in $\mathscr{B}_h(H)$ is a convex cone in $\mathscr{B}_h(H)$ which defines the order "$\leq$" on $\mathscr{B}_h(H)$. This convex cone is denoted with $\mathscr{B}^+(H)$. The set of all strictly positive (or positive invertible) operators in $\mathscr{B}_h(H)$ is denoted with $\mathscr{B}^{++}(H)$.

The *continuous functional calculus* is based on the Gelfand mapping Φ which is a *-isometric isomorphism from the space $C(\sigma(A))$ of all continuous functions that act on the spectrum $\sigma(A)$ of a self-adjoint operator A on H, onto the C*-algebra $C^*(A)$ generated with A and the identity. This mapping has the following properties:

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$,

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$,

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \sigma(A)} |f(t)|$,

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$,

$f, g \in C(\sigma(A))$, $\alpha, \beta \in \mathbb{C}$.

Thus the continuous functional calculus

$$f(A) = \Phi(f) \tag{1.39}$$

provides acting of the function $f \in C(\sigma(A))$ on the self-adjoint operator A itself.

In that sense, if A is a positive operator and $f_{1/2}(t) = \sqrt{t}$, then $A^{1/2} = f_{1/2}(A)$.

Furthermore, if A is a self-adjoint operator and f is a real valued continuous function defined on $\sigma(A)$ such that $f(t) \geq 0$, for all $t \in \sigma(A)$, then $f(A) \geq 0$, i.e. $f(A)$ is a positive operator.

The following order preserving property is a consequence of the continuous functional calculus.

Lemma 1.1 *Let $A \in \mathscr{B}_h(H)$ and let $f, g \in C(\sigma(A))$.*

$$\text{If } f(t) \geq g(t), \text{ for all } t \in \sigma(A), \text{ then } f(A) \geq g(A). \tag{1.40}$$

Additionally, $f(A) = g(A)$ if and only if $f(t) = g(t)$, for all $t \in \sigma(A)$.