Chapter 2

The weighted energy inequalities for convex functions

2.1 The weighted square integral inequalities for the first derivative of the function of a real variable

We consider the pair of twice continuously differential functions f and g defined on the closed bounded interval $[a,b]$. We assume that the function g is convex and the following requirement is satisfied:

$$|f''(x)| \leq g''(x), \quad a \leq x \leq b. \tag{2.1}$$

Let us introduce a family of nonnegative twice continuously differentiable weight functions $H : [a,b] \to \mathbb{R}$ which satisfy the following conditions

$$H(a) = H(b) = 0, \quad H'(a) = H'(b) = 0. \tag{2.2}$$

Theorem 2.1 *Let $f,\ g : [a,b] \to \mathbb{R}$ be two twice continuously differentiable functions which satisfy the requirement* (2.1) *and let $H : [a,b] \to \mathbb{R}$ be arbitrary nonnegative weight* *function such that condition* (2.2) *is fulfilled. Then the following inequality is valid*

$$\int_a^b \left(f'(x)\right)^2 H(x)dx \le \int_a^b \left[\left(\frac{f(x)}{2}\right)^2 + \left(\sup_{a\le t\le b} |f(t)|\right) g(x)\right] |H''(x)| dx. \tag{2.3}$$

Proof. Using the integration by parts

$$\begin{aligned}\int_a^b \left(f'(x)\right)^2 H(x)dx &= f(x)f'(x)H(x)\Big|_a^b - \int_a^b (f'H)'(x)f(x)dx \\ &= -\int_a^b f(x)f'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx \\ &= -\frac{1}{2}\int_a^b (f^2)'(x)H'(x)dx - \int_a^b f(x)f''(x)H(x)dx. \end{aligned} \tag{2.4}$$

Similarly, using $H'(a) = H'(b) = 0$,

$$\int_a^b (f^2)'(x)H'(x)dx = -\int_a^b f^2(x)H''(x)dx.$$

Now (2.4) becomes

$$\begin{aligned}\int_a^b \left(f'(x)\right)^2 H(x)dx &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx - \int_a^b f(x)f''(x)H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \int_a^b |f(x)||f''(x)|H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)| \int_a^b |f''(x)|H(x)dx \\ &\le \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)| \int_a^b g''(x)H(x)dx \\ \text{(repeated int. by parts)} &= \frac{1}{2}\int_a^b f^2(x)H''(x)dx + \sup_{a\le t\le b}|f(t)| \int_a^b g(x)H''(x)dx. \end{aligned}$$

□

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Corollary 2.1 *Under the same conditions as in the Theorem 2.1, the following bound is valid*

$$\int_a^b \left(f'(x)\right)^2 H(x)dx \le \|f\|_\infty \left(\frac{1}{2}\|f\|_p + \|g\|_p\right)\|H''\|_q \tag{2.5}$$

where $1 \le p \le \infty$, and p and q are conjugate exponents.

Proof. We apply Hölder inequality to the right-hand side of estimate (2.3).

Remark 2.1 *Let us notice that dominance (2.1) is equivalent to the existence of decomposition of the function f as the difference of two twice continuously differentiable convex functions, f_1 and f_2, such that, $f(x) = f_1(x) - f_2(x)$, $a \le x \le b$ and $g(x) = f_1(x) + f_2(x)$. Indeed, $|f''(x)| \le g''(x)$ is equivalent $-g''(x) \le f''(x) \le g''(x)$, that is,*

$$f''(x) + g''(x) \ge 0, \quad g''(x) - f''(x) \ge 0.$$

The latter means that the functions

$$f_1(x) = \frac{1}{2}(f(x) + g(x)), \qquad f_2(x) = \frac{1}{2}(g(x) - f(x))$$

are convex functions such that

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = f_1(x) + f_2(x). \tag{2.6}$$

Conversely, if f_1 and f_2 are two twice continuously differentiable convex such that (2.6) is valid, then it is obvious that we have dominance (2.1).

This remark suggests to write inequality (2.5) in a different form:

$$\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \le \|f_1 - f_2\|_\infty \left[\frac{1}{2}\|f_1 - f_2\|_p + \|f_1 + f_2\|_p\right] \|H''\|_q, \tag{2.7}$$

where $1 \le p \le \infty$.

Corollary 2.2 *Let f_1 and f_2 be twice continuously differentiable convex functions defined on a closed bounded interval $[a,b]$ and let the weight function H be equal to*

$$H(x) = (x-a)^2(b-x)^2, \quad a \le x \le b.$$

Then the following estimate holds

$$\int_a^b (f_1'(x) - f_2'(x))^2 H(x)dx \le \|f_1 - f_2\|_\infty [\frac{4\sqrt{3}}{9}\|f_1 + f_2\|_\infty + \frac{2\sqrt{3}}{9}\|f_1 - f_2\|_\infty](b-a)^3. \tag{2.8}$$

Proof. We have

$$H''(x) = 12x^2 - 12(a+b)x + 2(a^2 + 4ab + b^2),$$

and then,

$$\int_a^b |H''(x)| = 2(b-a)^3 \int_0^1 |6u^2 - 6u + 1| du = \frac{4\sqrt{3}}{9}(b-a)^3.$$

Finally, taking into account the latter expression in estimate (2.7), we come to the desired inequality (2.8). □

Remark 2.2 *Comparing the result stated in Corollary 2.2 with Theorem 2.1 from K. Shashiashvili and M. Shashiashvili [50], we come to the conclusion that the constant factor* $\frac{4\sqrt{3}}{9}$ *is twice less than the constant factor obtained in the latter paper.*

2.1.1 The weighted square integral estimates for the difference of derivatives of two convex functions

Now we consider two arbitrary bounded convex functions f and g on an infinite interval $[0,\infty)$. It is well known that they are continuous and have finite left and right hand derivatives $f'(x-)$, $f'(x+)$ and $g'(x-)$, $g'(x+)$ inside the open interval $(0,\infty)$. We will assume that there exists a positive number A such that if $x \geq A$, we have

$$|f'(x-)| \leq C, \quad |g'(x-)| \leq C \tag{2.9}$$

where C is a certain positive constant.
Let us assume also that the difference of the functions f and g is bounded on the infinite interval $[0,\infty)$:

$$\sup_{x \geq 0} |f(x) - g(x)| < \infty. \tag{2.10}$$

Introduce now the family of nonnegative twice continuously differentiable weight functions $H(x)$ defined on the open interval $(0,\infty)$, which satisfy the following conditions:

$$\lim_{x \to 0+} H(x) = 0, \quad \lim_{x \to \infty} H(x) = 0, \quad \lim_{x \to 0+} H'(x) = 0, \quad \lim_{x \to \infty} H'(x) = 0, \tag{2.11}$$

and

$$\int_0^\infty (|f(x)| + |g(x)|)|H''(x)| dx < \infty. \tag{2.12}$$

Theorem 2.2 *For arbitrary bounded convex functions f and g defined on* $[0,\infty)$ *satisfying conditions* (2.9) *and* (2.10) *and for any nonnegative twice continuously differentiable* *weight function H,* $0 < x < \infty$*, which satisfy conditions* (2.11) *and* (2.12)*, the following energy estimate is valid:*

$$\int_0^\infty (f'(x-) - g'(x-))^2 H(x)dx \le \frac{3}{2} \sup_{x\ge 0} |f(x) - g(x)| \int_0^\infty (|f(x)| + |g(x)|)|H''(x)|dx. \quad (2.13)$$

Proof. We will prove the theorem in two stages. In the first stage, we verify the validity of the statement for twice continuously differentiable convex functions satisfying conditions (2.9) and (2.10), and on second stage we approximate arbitrary convex functions satisfying the same conditions by smooth ones inside the interval $(0,\infty)$ in an appropriate manner. Afterwards we will pass with a limit in the previously established estimate.
Let the function F be defined as

$$F(x) = f(x) - g(x) \qquad 0 \le x < \infty.$$

Then F is twice continuously differentiable inside the infinite interval $(0,\infty)$ and at point zero, it has finite limit $F(0+)$.
Consider the following integral on a finite interval $[\delta, b]$ and use in it the integration by parts formula (here δ and b are arbitrary strictly positive numbers),

$$\int_\delta^b F'(x)(FH)'(x)dx = F'(x)F(x)H(x)\Big|_\delta^b - \int_\delta^b F''(x)(F(x)H(x))dx$$
$$= F(b)F'(b)H(b) - F(\delta)F'(\delta)H(\delta) - \int_\delta^b F''(x)F(x)H(x)dx. \quad (2.14)$$

The absolute value of the last integral

$$\left|\int_\delta^b F''(x)F(x)H(x)dx\right| \le \sup_{\delta\le x\le b} |F(x)| \int_\delta^b |f''(x) - g''(x)|H(x)dx$$
$$\le \sup_{\delta\le x\le b} |F(x)| \int_\delta^b (f''(x) + g''(x))H(x)dx \quad (2.15)$$

since $f''(x) \ge 0, \quad g''(x) \ge 0$, for $0 < x < \infty$.
Transforming the integral on the right-hand side of inequality (2.15),

$$\int_\delta^b (f''(x) + g''(x))H(x)dx = (f'(x) + g'(x))H(x)\Big|_\delta^b - \int_\delta^b (f'(x) + g'(x))H'(x)dx$$
$$= (f'(x) + g'(x))H(x)\Big|_\delta^b - (f(x) + g(x))H'(x)\Big|_\delta^b + \int_\delta^b (f(x) + g(x))H''(x)dx.$$

If we substitute the above expression in inequality (2.15), we obtain the estimate

$$\begin{aligned}\left|\int_{\delta}^{b} F''(x)F(x)H(x)dx\right| \leq & \sup_{\delta\leq x\leq b}|F(x)|\Big\{|f'(b)+g'(b)|H(b)\\ &+ |f'(\delta)+g'(\delta)|H(\delta)+|f(b)+g(b)||H'(b)|\\ &+ \int_{\delta}^{b}|f(x)+g(x)||H''(x)|dx\Big\}.\end{aligned}$$

Thus, from equality (2.14), we come to the following bound:

$$\begin{aligned}\left|\int_{\delta}^{b} F'(x)(FH)'(x)dx\right| \leq & |F(b)F'(b)|H(b)+|F(\delta)F'(\delta)|H(\delta)\\ &+\sup_{\delta\leq x\leq b}|F(x)|\Big\{|f'(b)+g'(b)|H(b)+|f'(\delta)+g'(\delta)|H(\delta)\\ &+|f(b)+g(b)||H'(b)|+\int_{\delta}^{b}|f(x)+g(x)|\cdot|H''(x)|dx\Big\}. \qquad (2.16)\end{aligned}$$

On the other hand, since

$$\int_{\delta}^{b} F'(x)(FH)'(x)dx=\int_{\delta}^{b}(F'(x))^2H(x)dx+\int_{\delta}^{b}F(x)F'(x)H'(x)dx,$$

we have

$$\begin{aligned}\int_{\delta}^{b}(F'(x))^2H(x)dx &= \int_{\delta}^{b}F'(x)(FH)'(x)dx-\frac{1}{2}\int_{\delta}^{b}(F^2)'(x)H'(x)dx\\ &= \int_{\delta}^{b}F'(x)(FH)'(x)dx-\frac{1}{2}\{F^2(x)H'(x)\Big|_{\delta}^{b}-\int_{\delta}^{b}F^2(x)H''(x)dx\}\\ &= \int_{\delta}^{b}F'(x)(FH)'(x)dx-\frac{1}{2}F^2(b)H'(b)|+\frac{1}{2}F^2(\delta)H'(\delta)dx+\frac{1}{2}\int_{\delta}^{b}F^2(x)H''(x)dx \qquad (2.17)\end{aligned}$$

Using inequality (2.16) in the expression (2.17), we arrive to the estimate

$$\int_{\delta}^{b}(F'(x))^2H(x)dx\leq\frac{1}{2}F^2(b)|H'(b)|+\frac{1}{2}F^2(\delta)|H'(\delta)|+|F(b)F'(b)|H^(b)$$

$$+|F^2(\delta)|F'(\delta)|H(\delta)+\sup_{\delta\le x\le b}|F(x)|\cdot\{\frac{3}{2}\int_\delta^b(|f(x)|+|g(x)|(|H)''(x)|dx)$$
$$+|f'(b)+g'(b)|H(b)+|f'(\delta)+g'(\delta)|H(\delta)$$
$$+|f(b)+g(b)||H'(b)|+|f(\delta)+g(\delta)H'(\delta)\}. \tag{2.18}$$

It is well known that any convex function is locally absolutely continuous (see, e.g., [59] Proposition 17 of Chapter 5) that is,

$$f(x_2)-f(x_1)=\int_{x_1}^{x_2}f'(u-)du,\quad 0<x_1<x_2<\infty. \tag{2.19}$$

As the lefthand derivative $f'(x-)$ of the convex function f is nondecreasing function, we have

$$f'(x_1-)\le f'(u-)\le f'(x_2-),\quad \text{if}\quad 0<x_1<u<x_2<\infty.$$

Therefore, from (2.19), we find that

$$f'(x_1-)(x_2-x_1)\le f(x_2)-f(x_1)\le f'(x_2-)(x_2-x_1), \tag{2.20}$$

where $0<x_1<x_2<\infty$.
Taking $x_1=x$, $x_2=2x$, we get

$$f'(x-)x\le f(2x)-f(x)\quad \text{for } x>0.$$

On the other hand, letting $x_1\searrow 0$ in inequality (2.20), we have

$$f(x_2)-f(0+)\le f'(x_2-)x_2,$$

that is,

$$f(x)-f(0+)\le f'(x-)x,\quad x>0.$$

Ultimately, we obtain the two-sided inequality

$$f(x)-f(0+)\le f'(x-)x\le f(2x)-f(x)\quad \text{for } x>0,$$

which gives (also for the function g)

$$\lim_{x\to 0+}xf'(x-)=0\quad \text{and}\quad \lim_{x\to 0+}xg'(x-)=0. \tag{2.21}$$

By equality (2.19) and using condition (2.9), we obtain the bound

$$|f(b)|\le|f(A)|\le C(b-A)\le|f(A)|+Cb\quad A\le b.$$

But since

$$|f(A)|\le\frac{|f(A)|}{A}b\quad \text{if } A\le b.$$

Therefore we can write, if $A \le b$

$$|f(b)H'(b)| \le |f(A)H'(b)| + Cb|H'(b)| \le \left(\frac{|f(A)|}{A} + C\right) b|H'(b)| \tag{2.22}$$

and similarly, if $A \le b$

$$|g(b)H'(b)| \le \left(\frac{|g(A)|}{A} + C\right) b|H'(b)| \quad \text{for } A \le b. \tag{2.23}$$

Using condition (2.11) and bounds (2.22) and (2.23), we get

$$\overline{\lim_{b\to\infty}} F^2(b)|H'(b)| \le \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f(b)+g(b)|)(|H'(b)|) = 0,$$

since

$$\overline{\lim_{b\to\infty}} (|f(b)+g(b)|)(|H'(b)|) = 0.$$

Moreover, from conditions (2.9) and (2.11), we find

$$\overline{\lim_{\delta\to 0+}} F^2(\delta)|H'(\delta)| = (|f(0+)-g(0+)|)^2 \overline{\lim_{\delta\to 0+}} |H'(\delta)| = 0,$$

$$\begin{aligned} \overline{\lim_{b\to\infty}} |F(b)F'(b-)|H(b) &\le \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} (|f'(b-)| + |g'(b-)|)(|H(b)|) \\ &\le 2C \sup_{0\le x<\infty} |F(x)| \overline{\lim_{b\to\infty}} H(b) = 0, \end{aligned} \tag{2.24}$$

$$\overline{\lim_{b\to\infty}} (|f'(b-)| + |g'(b-)|)(|H(b)|) \le 2C \overline{\lim_{b\to\infty}} H(b) = 0,$$

$$\overline{\lim_{\delta\to 0+}} |f(\delta)+g(\delta)||H'(\delta)| = |(f(0+)+g(0+))| \overline{\lim_{\delta\to 0+}} |H'(\delta)| = 0.$$

Using the mean value theorem, we have

$$\frac{H(\delta)}{\delta} = \frac{H(\delta)-H(0+)}{\delta} = H'(\upsilon_\delta), \quad \text{where } 0 < \upsilon_\delta < \delta,$$

therefore from condition (2.11), we deduce

$$\overline{\lim_{\delta\to 0+}} \frac{H(\delta)}{\delta} = 0. \tag{2.25}$$

Using the limit relations above and (2.21), we find

$$\overline{\lim_{\delta\to 0+}} |F(\delta)F'(\delta-)|H(\delta) \le \sup_{0\le x<\infty} |F(x)|) \overline{\lim_{\delta\to 0+}} |f'(\delta-) - g'(\delta-)|H(\delta)$$

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$$\leq \sup_{0\leq x<\infty}|F(x)|)\overline{\lim_{\delta\to 0+}}\left(|\delta f'(\delta-)|\frac{H(\delta)}{\delta}+|\delta g'(\delta-)|\frac{H(\delta)}{\delta}\right)=0, \tag{2.26}$$

and similarly

$$\overline{\lim_{\delta\to 0+}}|f'(\delta-)+g'(\delta-)|H(\delta)=0. \tag{2.27}$$

Now, in inequality (2.18), we pass with limit when $b\to\infty$ and $\delta\to 0$. Obviously, the left-hand side of the inequality increases and the right-hand side is bounded, when $b\to\infty$, $\delta\to 0$, therefore the left-hand side also converges to finite limit, so we come to the required estimate (2.13).

Next we move to the second stage of the proof. Consider two arbitrary convex functions f and g defined on $[0,\infty)$, satisfying conditions (2.9) and (2.10). We have to construct the sequences of twice continuously differentiable (in the open interval $(0,\infty)$) convex functions f_n and g_n approximating, respectively, the functions f and g inside the interval $[0,\infty)$ in an appropriate manner. To construct such sequences, we will use the following smoothing function:

$$\rho(x)=\begin{cases} C\exp[\frac{1}{x(x-2)}]; & 0<x<2,\\ 0; & \text{otherwise,}\end{cases}$$

where the factor C is chosen to satisfy the equality

$$\int_0^2 \rho(x)dx=1.$$

Define for $x\in[0,\infty),\ n\in\mathbb{N}$

$$f_n(x)=\int_0^\infty n\rho(n(x-y))f(y)dy,$$

$$g_n(x)=\int_0^\infty n\rho(n(x-y))g(y)dy. \tag{2.28}$$

For arbitrary fixed $\delta>0$ consider the restriction of functions f_n and g_n on the interval $[\delta,b]$ and let $n\geq 4/\delta$. Then $nx\geq 4$ for $x\in[\delta,b]$.

After we perform in (2.28) the change of variable $z=n(x-y)$, then we find

$$f_n(x)=\int_{-\infty}^{nx}\rho(z)f\left(x-\frac{z}{n}\right)dz,$$

$$g_n(x)=\int_{-\infty}^{nx}\rho(z)g\left(x-\frac{z}{n}\right)dz,$$

Since the function ρ is equal to zero outside the interval $(0,2)$, we can write

$$f_n(x) = \int_0^2 \rho(z) f\left(x - \frac{z}{n}\right) dz,$$
$$g_n(x) = \int_0^2 \rho(z) g\left(x - \frac{z}{n}\right) dz, \tag{2.29}$$

if $x \in [\delta, b], n \geq 4/\delta$.
From definition (2.28), it is obvious that the functions f_n and g_n are infinitely differentiable, while their convexity follows from the expressions (2.29).
Now we show the uniform convergence of the sequence of functions f_n to the function f on the interval $[\delta, b]$ (similarly, the uniform convergence of g_n to g). For this purpose, we use the uniform continuity of the function f on the interval $[\frac{\delta}{2}, b]$. For fixed $\varepsilon > 0$ there exists $\widehat{\delta} > 0$ such that we have

$$|f(x_2) - f(x_1)| \leq \varepsilon \quad \text{if } |x_2 - x_1| < \widehat{\delta}, \quad x_1, x_2 \in \left[\frac{\delta}{2}, b\right].$$

Take $n \geq \max\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\}$. Then for $0 \leq z \leq 2$ and $x \in [\delta, b]$, we get

$$\frac{z}{n} \leq \min\left\{\frac{\delta}{\widehat{\delta}}\right\}, \qquad x - \frac{z}{n} \geq \frac{\delta}{2}.$$

Hence

$$\left| f\left(x - \frac{z}{n}\right) - f(x) \right| \leq \varepsilon \quad \text{for } n \geq \max\left\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\right\}$$

and consequently

$$|f_n(x) - f(x)| = \left| \int_0^2 \rho(z) \left(f\left(x - \frac{z}{n}\right) - f(x) \right) dz \right| \leq \varepsilon \tag{2.30}$$

$$\text{for } \in [\delta, b] \text{ and } n \geq \max\left\{\frac{4}{\delta}, \frac{4}{\widehat{\delta}}\right\}. \tag{2.31}$$

Next we need to differentiate (2.29). For this purpose, we will use the following inequality ([18], page 114) concerning convex function $f(x)$ and its left-derivative $f'(x-)$

$$f'(x_1-) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2-), \qquad 0 < x_1 < x_2 < \infty.$$

Now, if we subsitute

$$x_1 = \left(x - \frac{z}{n}\right) - h, \quad x_2 = x - \frac{z}{n},$$

where $0 < h < \frac{\delta}{4}$, we have

$$f'\left(\left((x - \frac{z}{n} - h\right) - \right) \leq \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \leq f'\left(\left((x - \frac{z}{n}\right) - \right)$$

for $x \in [\delta, b],\ 0 \leq z \leq 2,\ 0 < h < \frac{\delta}{4}$, and $n \geq \frac{4}{\delta}$.
It is well known that the left derivative of the convex function is nondecreasing and, since,

$$x - \frac{z}{n} - h \geq \frac{\delta}{4}, \qquad x - \frac{z}{n} \leq b$$

we can write

$$f'\left(\frac{\delta}{4} - \right) \leq \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \leq f'(b-).$$

This shows that the family of functions

$$\Phi_h^{n,x}(z) = \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h}$$

is uniformly bounded by the constant $D = |f'(b-)| + |f'(\frac{\delta}{4}-)|$ if $x \in [\delta, b],\ 0 \leq z \leq 2,\ 0 < h < \frac{\delta}{4}$, and $n \geq (\frac{4}{\delta})$.
Using expression (2.29), we can write

$$\frac{f_n(x) - f_n(x-h)}{h} = \int_0^2 \rho(z) \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} dz.$$

Taking limit as h tends to zero and using dominated convergence theorem, we obtain the formula

$$f_n'(x) = \int_0^2 \rho(z) f'\left(\left(x - \frac{z}{n} - \right)\right) dz \tag{2.32}$$

for $x \in [\delta, b]$ and $n \geq \frac{4}{\delta}$.
Using (2.32) let us show that for fixed $x \in [\delta, b]$, the sequence $f_n'(x)$ converges to the left-derivative $f'(x-)$.
We have

$$f_n'(x) - f'(x-) = \int_0^2 \rho(z) \left(f'\left(\left(x - \frac{z}{n} - \right)\right) - f'(x-) \right) dz, \tag{2.33}$$

where $n \geq \frac{4}{\delta}$. Choose arbitrary $\varepsilon > 0$. Since the left-derivative $f'(x-)$ is left continuous, we can find $N(\varepsilon)$ such that (for $0 \leq z \leq 2$):

$$\left| f'\left(\left(x - \frac{z}{n} - \right)\right) - f'(x-) \right| \leq \varepsilon \quad \text{if } n \geq N(\varepsilon).$$

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Then we have

$$f_n'(x) - f'(x-) = \int_0^2 \rho(z)\varepsilon dz = \varepsilon \qquad \text{if } x \in [\delta, b], \quad n \geq \max\left\{\frac{4}{\delta'}, N(\varepsilon)\right\},$$

that is,

$$\lim_{n\to\infty} f_n'(x) = f'(x-), \quad x \in [\delta, b].$$

Similarly,

$$\lim_{n\to\infty} g_n'(x) = g'(x-) \text{if } x \in [\delta, b]. \tag{2.34}$$

Now we apply (2.18) estimate for the function $F_n(x) = f_n(x) - g_n(x)$ on $[\delta, b]$,

$$\begin{aligned}
\int_\delta^b (F_n'(x))^2 H(x)dx \leq\ & \frac{1}{2}F_n^2(b)|H'(b)| + \frac{1}{2}F_n^2(\delta)|H'(\delta)| + |F_n(b)F_n'(b)|H^(b) \\
&+ |F_n^2(\delta)|F_n'(\delta)|H(\delta) + \sup_{\delta\leq x\leq b} |F_n(x)| \\
&\times \{\frac{3}{2}\int_\delta^b (|f_n(x)| + |g_n(x)|(|H)''(x)|dx) + |f_n'(b) + g_n'(b)|H(b) \\
&+ |f_n'(\delta) + g_n'(\delta)|H(\delta) + |f_n(b) + g_n(b)||H'(b)| \\
&+ |f_n(\delta) + g_n(\delta)H'(\delta)\}.
\end{aligned} \tag{2.35}$$

For $x \in [\delta, b],\ 0 \leq z \leq 2$ and $n \geq \frac{4}{\delta}$, we have

$$f'\left(\frac{\delta}{2}-\right) \leq f'\left(\left(x - \frac{z}{n} -\right)\right) \leq f'(b-).$$

Multiplying this inequality by $\rho(z)$ and integrating by z over $(0,2)$ using (2.32), we have

$$f'\left(\frac{\delta}{2}-\right) \leq f_n'(x) \leq f'(b-),$$

and then

$$|f_n'(x)| \leq |f'(b-)| + \left|f'\left(\frac{\delta}{2}-\right)\right|, \quad \text{if } x \in [\delta, b],\ n \geq \frac{4}{\delta}.$$

Similarly, for the functions $g_n(x)$, we have

$$|g_n'(x)| \leq |g'(b-)| + \left|g'\left(\frac{\delta}{2}-\right)\right|,$$

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From the latter bounds, we obtain

$$|F_n'(x)| \le |f'(b-)| + |g'(b-)| + \left|f'\left(\frac{\delta}{2}-\right)\right| + \left|g'\left(\frac{\delta}{2}-\right)\right|$$

if $x \in [\delta, b]$ and $n \ge \frac{4}{\delta}$.
Hence the sequence of the functions F_n' is uniformly bounded on the interval $[\delta, b]$ for $n \ge \frac{4}{\delta}$. Thus we can apply the bounded convergence theorem in the left-hand side of inequality (2.35). Letting n to infinity, we will have

$$\begin{aligned}\int_\delta^b (F'(x-))^2 H(x)dx \le\ & \frac{1}{2}F^2(b)|H'(b)| + \frac{1}{2}F^2(\delta)|H'(\delta)| + |F(b)F'(b-)|H^(b) \\ &+ |F^2(\delta)|F'(\delta-)|H(\delta) + \\ +\|f\|_{L^\infty} \times \{ & \frac{3}{2}\int_\delta^b (|f(x)| + |g(x)|(|H)''(x)|dx) + |f'(b-) + g'(b-)|H(b) \\ &+ |f'(\delta-) + g'(\delta-)|H(\delta) + |f(b) + g(b)||H'(b)| \\ &+ |f(\delta) + g(\delta)H'(\delta)\}. \end{aligned} \tag{2.36}$$

The left-hand side of inequality (2.36) obviously increases when $b \to \infty$ and $\delta \to 0$ and the right-hand side is bounded by the assumption (2.12) and the limit relations (2.24)-(2.27). Therefore passing onto limit $b \to \infty$ and $\delta \to 0$ in inequality (2.36), we arrive to the desired estimate (2.13). □