1. Introduction

This work contains a compilation of a portion of the research that has been done in the areas of extension theory, dimension theory, and connections between these two and shape theory [44]. The main emphasis will be on the former two, but we shall include the third because of the way it has been intertwined with the others.

When we found it possible, we made steps to improve the efficiency and clarity of what had been published previously in the literature relative to these subjects. So the reader might even find some new results in the following even though it was not the authors' intent to produce a research article. This writing is not meant to be a historical presentation, but to set the tone, we have provided, in the ensuing paragraphs, at least a rudimentary scan of the events leading to our survey.

Dimension theory began in the early part of the 20th century when Henri Poincaré observed in the course of a discussion of space of 3-dimensions, that nobody could explain what this notion meant (see p. 3 of [31]). One assumes that he wanted a topological definition of dimension, which, of course, did not exist at that time. Poincaré had some good ideas of how to define dimension, but he died before having the opportunity to put them into play. The fundamental problem was to find a way to assign a topologically invariant non negative integral value n to a space that would represent its "dimensional thickness."

It could have been the influence of set theory as advanced by Cantor which really set out this line of thinking. By his approach, one could not distinguish $\mathbb{R}$ from $\mathbb{R}^2$ as sets. Moreover, in those times, if one spoke of the notion of an n-dimensional object it was simply thought to be something requiring n continuous parameters for its expression. This was naive; already the concept of a space-filling curve as described by Peano was in the air. Putting it another way, since it was possible to fill a disk in the plane, or analogously, a ball in space, with a continuous 1-parameter curve, then how could such a precept be used to quantify n-dimensional space?

By the early 1920's (to find references that are more historical, one might consult [25], [26], [24] or [77]), Poincaré's call for a theory of dimension had been answered. Work of L. Brouwer, which gave an early start but seems to have been overlooked in any case, was in a short time superseded by that of Menger and Urysohn who did their research independently of each other. It is their theory of dimension which forms the basis for the one we use today. When we later discuss the various approaches to defining dimension, one will be surprised at the several different, but equivalent (and useful) formulations of these definitions which have arisen over the years.

In the late 1920's and early 1930's, P. S. Alexandroff introduced a new approach to dimension. At first it was one based on the group $\mathbb{Z}$ of integers and was referred to as homological dimension theory. Nowadays we recognize that such a theory can be defined for any abelian group G via Čech cohomology, and this type of dimension is called cohomological dimension modulo G. Thus cohomological dimension theory is really a whole spectrum of dimension theories, one for each algebraically distinct abelian group.

Among the collection of definitions, both for dimension and for cohomological dimension, it turned out that an approach based on extension of an arbitrary map from a closed subset of a space–with range in a certain type of complex–to a map of the entire space to that complex, could be used to define all the above-mentioned types of dimension. This was recognized and used prominently in the early 1980's in work of J. Walsh [103], where he published a proof of the important Edwards-Walsh cell-like resolution theorem. There the comparison was between dimension and cohomological dimension modulo $\mathbb{Z}$. But soon others were to see that the entire universe of dimension theories should be viewed in one consolidated way, and a new subject arose.

This new area, called extension theory, (originated by A. N. Dranishnikov, [17]) serves as an umbrella over all the known dimension theories and therefore allows a certain efficiency which was not known previously. It covers more ground than the others do, and we shall see that it has an application even in conjunction with shape theory.

Our first priority will be to review and explain the classical definitions and facts from dimension and to give at least some introduction to the notion of cohomological dimension. Then we will introduce the concept of extension theory. We shall give in detail some proofs of basic theorems in that area.

In a study such as ours, one must always be concerned with which class of topological spaces is most suitable for the development of a coherent and broadly based theory. Although we will try to make definitions that apply to topological spaces in general, it will not be feasible to develop a theory that covers such a broad range of possibilities. Hence many times we will restrict to classes such as paracompact spaces, stratifiable spaces (a generalization of metrizable spaces), metrizable spaces, compact Hausdorff spaces, or even compact metrizable spaces. Structures such as CW-complexes and triangulated polyhedra will play a prominent role in this theory. We shall also place much emphasis on the preservation of dimension-theoretic or extension-theoretic properties in the limits of inverse sequences and systems, direct sequences and systems, and generalized inverse sequences called semi-sequences [34], [36]. Whether a dimension-theoretic or extension-theoretic property survives in all subspaces (strong inheritance) or only in all closed subspaces (weak inheritance) will be treated where known.

Throughout this presentation, *map* will mean continuous function.

2. Primitive Dimension Structure

Definition 2.1. Let $\mathscr{C}$ be a class of spaces that includes $\emptyset$. A **primitive dimension structure** for $\mathscr{C}$ is a subset $\mathscr{D} \subset \mathscr{C} \times \mathbb{Z}_{\geq -1}$ such that,

(i) $(X,-1) \in \mathscr{D}$ if and only if $X = \emptyset$, and

(ii) if $X \in \mathscr{C}, m \in \mathbb{Z}_{\geq -1}$, and $(X,m) \in \mathscr{D}$, then $(X,m+1) \in \mathscr{D}$.

Definition 2.2. Let $\mathscr{C}$ be a class of spaces and $\mathscr{D}$ a primitive dimension structure for $\mathscr{C}$. We define the **induced dimension function** $D_{\mathscr{D}} : \mathscr{C} \to \mathbb{Z}_{\geq -1} \cup \{\infty\}$ as follows:

(i) If $m \in \mathbb{Z}_{\geq -1}$, $(X,m) \in \mathscr{D}$, and m is the first element of $\mathbb{Z}_{\geq -1}$ with $(X,m) \in \mathscr{D}$, then $D_{\mathscr{D}}(X) = m$;

(ii) $D_{\mathscr{D}}(X) = \infty$ otherwise.

We often write $D_{\mathscr{D}}(X) \leq m$ when $D_{\mathscr{D}}(X) = n$ and $n \leq m$, even in case $m = \infty$.

Lemma 2.3. *If $\mathscr{C}$ is a class of spaces and $\mathscr{D}$ is a primitive dimension structure for $\mathscr{C}$, then for all $X \in \mathscr{C}, D_{\mathscr{D}}(X) = -1$ if and only if $X = \emptyset$.*

Our first encounter with a primitive dimension structure will yield both small and large inductive dimension for the class of topological spaces.

Definition 2.4. Let (A,B) be a disjoint pair of subsets of a space X. A closed subset S of X is called a **separator** (sometimes **partition**) of (A,B) if $X \setminus S$ can be written as $U \cup V$ where U,V are open in $X, A \subset U, B \subset V$, and $U \cap V = \emptyset$.

Clearly a separator of (A,B) is also a separator of (B,A).

Definition 2.5. Let $\mathscr{F}$ be a collection of disjoint pairs of closed subsets of a space X. Then $\mathscr{F}$ is called an **essential family** in X if for each collection $\mathscr{G} = \{G_F \mid F \in \mathscr{F}\}$ where for each $F \in \mathscr{F}$, G_F is a separator of F, it is true that $\bigcap\{G_F \mid F \in \mathscr{F}\} \neq \emptyset$.

For (D1)–(D6), let $\mathscr{C}$ be the class of topological spaces.

(D1) Small inductive dimension, ind: Define $\mathscr{D}_{-1} = \{(\emptyset, -1)\}$. Suppose that $m \in \mathbb{Z}_{\geq 0}$ and we have defined $\mathscr{D}_n$ for all $X \in \mathscr{C}$ and $n < m$. Then we put $(X,m) \in \mathscr{D}_m$ if $X \neq \emptyset$ and for each closed subset $A \subset X$ and $x \in X \setminus A$, there is a separator S of $(\{x\},A)$ such that $(S, m-1) \in \mathscr{D}_{m-1}$. Let $\mathscr{D} = \bigcup\{D_m \mid m \in \mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}} = \text{ind}$.

(D2) Large inductive dimension, Ind: Large inductive dimension Ind can be obtained in a similar way. Let $\mathscr{D}_{-1} = \{(\emptyset, -1)\}$. Suppose that $m \in \mathbb{Z}_{\geq 0}$ and we have defined $\mathscr{D}_n$ for all $X \in \mathscr{C}$ and $n < m$. Then we put $(X,m) \in \mathscr{D}_m$ if $X \neq \emptyset$ and for each disjoint pair (A,B) of closed subsets of X, there is a separator S of (A,B) such that $(S, m-1) \in \mathscr{D}_{m-1}$. Let $\mathscr{D} = \bigcup\{D_m \mid m \in \mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}} = \text{Ind}$.

(D3) Separator dimension, sep-dim: Consult Definition 4.4 of [73] for the ensuing idea. Let $\mathscr{D}_{-1} = \{(\emptyset, -1)\}$. Suppose $X \in \mathscr{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Then we put $(X,m) \in \mathscr{D}_m$ if $X \neq \emptyset$ and X has an essential family of cardinality $\leq m$. Let $\mathscr{D} = \bigcup\{\mathscr{D}_m \mid m \in \mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}}$=sep-dim.

Definition 2.6. A space X is called **weakly infinite-dimensional** provided sep-dim$(X) = \infty$ and X does not have an infinite essential family. We call X **strongly infinite-dimensional** if sep-dim$(X) = \infty$ and X has an infinite essential family.

(D4) Cohomological dimension, $\dim_G$: Let G be an Abelian group. We will define the G-**cohomological dimension** of a space X, $\dim_G X$, as follows. Let $\mathscr{D}_{-1} = \{(\emptyset, -1)\}$, suppose that $X \in \mathscr{C}$, $X \neq \emptyset$, and $m \in \mathbb{Z}_{\geq 0}$. Then we put $(X,m) \in \mathscr{D}_m$ if there exists $n \in \mathbb{N}$ such that $\check{H}^n(X,A;G) = 0$ for all closed subsets A of X and m is the first element of $\mathbb{Z}_{\geq 0}$

such that $\check{H}^{m+1}(X,A;G)=0$ for all closed subsets A of X. Put $\mathscr{D}=\bigcup\{\mathscr{D}_m \mid m\in\mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}}=\dim_G$.

(D5) Covering dimension, dim: Covering dimension dim is managed as follows. Put $\mathscr{D}_{-1}=\{(\emptyset,-1)\}$. Suppose that $m\in\mathbb{Z}_{\geq 0}, X\in\mathscr{C}$, and $X\neq\emptyset$. Then $(X,m)\in\mathscr{D}_m$ if each open cover of X has an open refinement of order $\leq m+1$ (i.e., each element of X lies in at most $m+1$ elements of the cover). Put $\mathscr{D}=\bigcup\{\mathscr{D}_m \mid m\in\mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}}=\dim$.

Before providing the next notion, we need a definition.

Definition 2.7. Let X and K be spaces. Then we write $X\tau K$ and say that X is an **absolute co-extensor** for K if for each closed subset A of X and each map $f: A\to K$, there exists a map of X to K that extends f. It is also said in this case that K is an **absolute extensor** for X. Indeed, if $\mathscr{C}$ is a class of spaces and for each $X\in\mathscr{C}$, $X\tau K$, then we shall say that K is an **absolute extensor** for $\mathscr{C}$, and for this we write $K\in\mathrm{AE}(\mathscr{C})$. In case the class $\mathscr{C}$ consists of just one space X, then we often write $K\in\mathrm{AE}(X)$.

Let us make a remark here concerning the logic in the notion of $X\tau K$. In case $X\neq\emptyset$ and $K=\emptyset$, then $X\tau K$ is impossible. When results are stated later, and it is not certain whether this phenomenon might occur, then we ask the reader to make accommodations for this special situation. But in almost all settings, one should assume that $K\neq\emptyset$.

(D6) Extension sequence dimension, $\mathscr{K}$-dim: Let $\mathscr{K}=\{K_i \mid i\in\mathbb{Z}_{\geq 0}\}$ be an indexed collection of spaces. We can define $\mathscr{K}$-dim in the following way. Put $\mathscr{D}_{-1}=\{(\emptyset,-1)\}$. Let $X\in\mathscr{C}, X\neq\emptyset$, and $m\in\mathbb{Z}_{\geq 0}$. We put $(X,m)\in\mathscr{D}_m$ if $X\tau K_m$. Let $\mathscr{D}=\bigcup\{\mathscr{D}_m \mid m\in\mathbb{Z}_{\geq -1}\}$. Then $D_{\mathscr{D}}=\mathscr{K}$-dim.

Two important examples of this ensue, but this time we must restrict the class of spaces. Let $\mathscr{C}$ be the class of paracompact Hausdorff spaces (paracompacta).

(D6A) If $\mathscr{K}=\{S^i \mid i\in\mathbb{Z}_{\geq 0}\}$, then $\mathscr{K}$-dim $=\dim$.

(D6B) If $\mathscr{K}=\{K(G,i) \mid i\in\mathbb{Z}_{\geq 0}\}$ where G is an abelian group, then $\mathscr{K}$-dim $=\dim_G$.

The spaces $K(G,i)$ in (D6B) are called Eilenberg-MacLane complexes, and will be discussed in Section 7.

3. Rudiments of Dimension Theory

In the early stages of the development of dimension theory, there were two versions of inductive dimension, small (ind) and large (Ind). It has become standard to use large inductive dimension, saving small inductive dimension for the special case (separable metrizable spaces) where it is most useful. Let us begin with a definition.

Definition 3.1. We say that a space X is **infinite-dimensional** if $\mathrm{Ind}X=\infty$. Otherwise we say that X is **finite-dimensional**.

It is difficult to believe that one can go very far with this definition of dimension. Let us see what we can do initially. We shall use $\mathbb{Q}$ and $\mathbb{I}$ respectively to designate the rationals and the irrationals as subspaces of $\mathbb{R}$.

Proposition 3.2. *The dimension function* Ind *satisfies the following.*

(i) *The empty set and only the empty set has* $\operatorname{Ind} = -1$.

(ii) *If X is a nonempty space with the discrete topology, then* $\operatorname{Ind} X = 0$.

(iii) *If X is a countably infinite metrizable space $($of necessity separable$)$, then* $\operatorname{Ind} X = 0$.

(iv) $\operatorname{Ind}\mathbb{Q} = 0 = \operatorname{Ind}\mathbb{I}$.

(v) *If X is a connected space of cardinality at least* 2, *then* $\operatorname{Ind} X \geq 1$.

(vi) $\operatorname{Ind}\mathbb{R} \geq 1$.

(vii) *If* $-1 \leq m \leq n \leq \infty$ *and* $\operatorname{Ind} X \leq m$, *then* $\operatorname{Ind} X \leq n$.

As a hint to proving Proposition 3.2(3), we would suggest that for a given disjoint pair A and B of closed subsets of X, apply the Urysohn lemma.

A lemma will help us prove the next theorem. As usual, we shall reserve I for the unit interval $[0,1]$.

Lemma 3.3. *Let D be a nonempty closed subset of I and U a neighborhood of D. Then there exists a finite, pairwise disjoint collection $\mathscr{F}$ of closed intervals of I such that* $D \subset \operatorname{int}_I \bigcup \mathscr{F} \subset \bigcup \mathscr{F} \subset U$.

Theorem 3.4. $\operatorname{Ind} I = 1$.

Proof. By an application of Proposition 3.2(5), $\operatorname{Ind} I \geq 1$. So we need to demonstrate the reverse inequality.

Let A and B be disjoint closed subsets of I and select a map $f : I \to I$ with $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. (Note that we are allowing for the possibility that A or B is empty.) Let $D = f^{-1}(\{\frac{1}{2}\})$; surely D is a separator of (A,B) in I.

If $D = \emptyset$, then define $S = D$; we see from Proposition 3.2(1) that $\operatorname{Ind} S \leq 0$, and of course that S is a separator of (A,B) in I.

Otherwise choose a neighborhood U of the closed subset D of I such that $(A \cup B) \cap U = \emptyset$, and select a collection $\mathscr{F}$ as in Lemma 3.3. Clearly $\bigcup \mathscr{F}$ is a separator of (A,B) in I. For each $J \in \mathscr{F}$, let $x_J \in J$, and put $S = \{x_J \mid J \in \mathscr{F}\}$. Then S is a finite, nonempty subset of I. By Proposition 3.2(2), $\operatorname{Ind} S = 0$. To complete our proof, it is sufficient to show that S is a separator of (A,B) in I.

If not, then there exist $a \in A$ and $b \in B$ such that $[a,b] \cap S = \emptyset$. There is $J \in \mathscr{F}$ such that $[a,b] \cap J \neq \emptyset$. But neither a nor b is in J, so $J \subset [a,b]$. This implies that $x_J \in [a,b]$ and hence $[a,b] \cap S \neq \emptyset$, a contradiction. □

The dimension of a topological space is a quantity which, intuitively, measures the “thickness” of that space in comparison with others. This accepted, one would think that it would automatically be the case that if $X \subset Y$, then $\operatorname{Ind} X \leq \operatorname{Ind} Y$. Unfortunately, this is not true in general; it does not even work for the class of paracompacta. We shall remark later about some other phenomena of dimension theory which are somewhat disagreeable. This is the price one has to pay in order to get a theory which conforms in the most important cases to the original purposes of the subject.

In the next section we shall begin to state some of the fundamental theorems of dimension theory. From them it will be possible to prove some significant theorems and perhaps to answer some questions which may have been raised because of what we have done so far.

4. Subspace and Sum Theorems

There are two important versions of the "Subspace Theorem" for Ind. Let us state the first and easiest one, the "Weak Subspace Theorem."

Proposition 4.1. (Weak Subspace Theorem) *Let A be a closed subspace of a space X. Then* $\operatorname{Ind} A \leq \operatorname{Ind} X$.

To get at the stronger one we need the concept of a strongly hereditarily normal space. This can be found on page 128 of [26].

Theorem 4.2. (STRONG SUBSPACE THEOREM) *Let A be a subspace of a strongly hereditarily normal space X. Then* $\operatorname{Ind} A \leq \operatorname{Ind} X$.

If one applies Theorem 2.1.4. of [26], which states that hereditarily paracompact spaces are strongly hereditarily normal, then we see that every metrizable space is strongly hereditarily normal. We get the following corollary.

Corollary 4.3. *The Strong Subspace Theorem holds true for metrizable spaces.*

Proposition 4.4. *For all $n \geq 1$,* $\operatorname{Ind} \mathbb{R}^n \geq 1$.

We know from 3.2(6) that $\operatorname{Ind} \mathbb{R} \geq 1$, but we are not yet able to conclude that $\operatorname{Ind} \mathbb{R} \leq 1$, and hence that $\operatorname{Ind} \mathbb{R} = 1$. Let us state a theorem which will provide the help needed to deduce this.

Theorem 4.5. (Sum Theorem) *If a hereditarily paracompact space X can be written as the countable union $\bigcup\{A_i \mid i \in \mathbb{N}\}$ of closed subspaces A_i where $\operatorname{Ind} A_i \leq n$ for each i, then* $\operatorname{Ind} X \leq n$.

By the way, this is not the strongest version of the Sum Theorem. One may find ([26], Theorem 2.3.11.) a stronger version of it.

Theorem 4.6. (Stronger Sum Theorem) *Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a σ-locally finite closed cover of a strongly hereditarily normal space X such that $\operatorname{Ind} F_\gamma \leq n$ for every $\gamma \in \Gamma$. Then* $\operatorname{Ind} X \leq n$.

We get the following corollary.

Corollary 4.7. *Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a σ-locally finite closed cover of a hereditarily paracompact space X such that* $\operatorname{Ind} F_\gamma \leq n$ *for every $\gamma \in \Gamma$. Then* $\operatorname{Ind} X \leq n$.

Proposition 4.8. $\operatorname{Ind}\mathbb{R} = 1$.

It turns out that when working with separable metrizable spaces it is often convenient to use small inductive dimension. Here is an important theorem in connection with three types of dimension.

Theorem 4.9. *Let X be a metrizable space*; *then*

(i) $\operatorname{Ind} X = \text{sep-dim} X$, *and*

(ii) $\operatorname{ind} X = \operatorname{Ind} X$ *if X is separable.*

It is false in general that ind and Ind agree for arbitrary metrizable spaces. There is a famous example due to Prabir Roy [85] which proves that these two dimension functions part company in the class of metrizable spaces. There is some discussion of this topic in [26].

Proposition 4.10. *If X is a subset of $\mathbb{R}$ which is nonempty and contains no interval, then* $\operatorname{Ind} X = 0$.

This might be misleading when one considers totally disconnected spaces.

Proposition 4.11. *If X is a separable, metrizable, locally compact and totally disconnected space, then* $\operatorname{Ind} X \leq 0$.

One should note that there is more than one definition of total disconnectedness. We may agree here to use the one that says that each component of X is singleton. Now it turns out that there are totally disconnected spaces of every dimension and even infinite-dimensional ones. This is another one of those anomalies of dimension theory, one that we have to live with in order to preserve a good theory.

We stated in the beginning that Poincaré wanted a theory of dimension in which "space" would be 3-dimensional, and, of course, all obviously n-dimensional spaces (e.g., $\mathbb{R}^n$) would have dimension n. Here is some warm-up.

Proposition 4.12. *The following are true.*

(i) If $X \subset \mathbb{R}^2$ is a rectangular disk, then $\operatorname{Ind}(\operatorname{bd}_{\mathbb{R}^2} X) = 1$.

(ii) $\operatorname{Ind}\mathbb{R}^2 \leq 2$.

(iii) For all $n \in \mathbb{N}$, $\operatorname{Ind}\mathbb{R}^n \leq n$.

It is a different story to prove that $\operatorname{Ind}\mathbb{R}^n = n$. This is frequently done using some tool from algebraic topology such as higher homotopy groups or homology groups. But there are other techniques that use only combinatorial methods such as that in [73] where it is proved that $I^n \cong I^m$ if and only if $n = m$. It is shown there that $\text{sep-dim}\, I^n = n$. With this and the Strong Subspace Theorem, one may conclude:

Theorem 4.13. *For all $n \in \mathbb{N}$,* $\operatorname{Ind}\mathbb{R}^n = n$.

We finish this section with some statements connecting Euclidean spaces and dimension theory.

Theorem 4.14. *Let $X \subset \mathbb{R}^n$; then $\operatorname{Ind}X = n$ if and only if the interior of X is nonempty.*

Theorem 4.15. *If $X \subset \mathbb{R}^n$ and X separates $\mathbb{R}^n$, then $\operatorname{Ind}X \geq n-1$.*

Theorem 4.16. (Theorem on Invariance of Domain) *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an embedding, then $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$.*

Proposition 4.17. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an injective map, then f is an embedding onto an open set.*

5. More Fundamentals

One theorem that is peculiar to dimension theory is the next one. We do not know of any analogous theorem in other theories (of extension or dimension).

Theorem 5.1. (Decomposition Theorem) *Let X be a metrizable space and $-1 \leq n < \infty$. Then $\operatorname{Ind}X \leq n$ if and only if X can be written as $\bigcup\{X_i \mid 0 \leq i \leq n\}$, where $\operatorname{Ind}X_i \leq 0$ for each $0 \leq i \leq n$.*

Theorem 5.2. (Menger-Urysohn Addition Theorem) *Let A and B be finite-dimensional subsets of a metrizable space X. Then $\operatorname{Ind}(A \cup B) \leq \operatorname{Ind}A + \operatorname{Ind}B + 1$.*

Why do we have to add 1 in the preceding statement? Just consider that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ while noting that $\operatorname{Ind}\mathbb{Q} = \operatorname{Ind}\mathbb{I} = 0$. In case the dimension of one of the summands in Theorem 5.2 is ∞, then the result would be rephrased to say that $\operatorname{Ind}(A \cup B) = \infty$.

When $n \geq 1$, one can give an explicit decomposition of $\mathbb{R}^n$ into $n+1$ 0-dimensional subspaces in the following manner. For $0 \leq k \leq n$, put Q_k^n equal the set of points in $\mathbb{R}^n$ having exactly k rational coordinates. It can be proved that $\operatorname{Ind}Q_k^n = 0$. Now note that $\mathbb{R}^n = \bigcup\{Q_k^n \mid 0 \leq k \leq n\}$.

Theorem 5.3. (Product Theorem) *Let X and Y be metrizable spaces. Then $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind}X + \operatorname{Ind}Y$.*

You may wonder why less than or equal; why not equality? There are examples to show that a "deficiency" may occur in the product. Namely, there is even a metrizable compactum X with $\operatorname{Ind}X = 2$, but $\operatorname{Ind}(X \times X) = 3$. This may be considered another anomaly of dimension theory.

Theorem 5.4. (Completion Theorem) *If X is a metrizable space, then there exist a completely metrizable space Y with* $\operatorname{Ind} Y = \operatorname{Ind} X$ *and an embedding of X into Y.*

Theorem 5.5. (Compactification Theorem) *For every separable metrizable space X, there exist a metrizable compactum Y with* $\operatorname{Ind} Y = \operatorname{Ind} X$ *and an embedding of X into Y.*

Definition 5.6. Let $\mathscr{F}$ be a class of spaces. A space X is called **universal** for $\mathscr{F}$ if X is in $\mathscr{F}$ and for every $Y \in \mathscr{F}$, there exists an embedding of Y into X.

Theorem 5.7. (Theorem on Universal Metrizable Compacta) *Let $-1 \leq n \leq \infty$. Put $\mathscr{F}_n$ equal the class of compact metrizable spaces of* $\operatorname{Ind} \leq n$*. Then there exists a space X which is universal for $\mathscr{F}_n$.*

6. Covering Dimension, Other Approaches

Checking the literature (e.g., [26]), one may discover that our definition of covering dimension is at odds with what one typically sees. The difference is that, standardly, the open covers are required to be finite. In the class of spaces of most interest to us, metrizable spaces (or even compact Hausdorff spaces), the two definitions are equivalent. Let us next quote Theorem 4.1.3 of [26], one of the most important theorems about the equivalence of distinct dimension notions.

Theorem 6.1. *For each metrizable space X,* $\operatorname{Ind} X = \dim X$.

Henceforward, when we speak of the "dimension" of a space we will mean its covering dimension dim unless otherwise specified. We noted in Section 2 that if $\mathscr{K} = \{S^i \,|\, i \in \mathbb{Z}_{\geq 0}\}$, then $\mathscr{K}$*-dimX* $= \dim X$ for all paracompacta X. Hence we have:

Theorem 6.2. *Let X be a paracompact space and n a nonnegative integer. Then* $\dim X \leq n$ *if and only if $X \tau S^n$.*

If one takes the point of view that the statement about extension of maps in Theorem 6.2 is the basis of the definition of dimension, then a whole new perspective comes into view. We shall see much more of this later. Let us now state some other items of interest in dimension theory, mainly for the class of metrizable spaces.

Theorem 6.3. *Let X be a metrizable space, and Y be a subspace of X with* $\dim Y \leq 0$*. Then for every disjoint pair (A,B) of closed subsets of X, there exists a separator C of (A,B) such that $C \cap Y = \emptyset$.*

Proposition 6.4. *Let X be a metrizable space, n be a nonnegative integer, and Y be a subspace of X with* $\dim Y \leq n$*. Then for every disjoint pair (A,B) of closed subsets of X, there exists a separator C of (A,B) such that* $\dim(C \cap Y) \leq n-1$.

Proposition 6.5. *Let X be a metrizable space, n be a nonnegative integer, and suppose that $\dim X \leq n$. Then for each collection $\{(A_i, B_i) \mid 1 \leq i \leq n+1\}$ of disjoint pairs of closed subsets of X, there exists a collection $\{C_i \mid 1 \leq i \leq n+1\}$ of respective separators of the pairs (A_i, B_i) such that $\bigcap\{C_i \mid 1 \leq i \leq n+1\} = \emptyset$.*

The Hilbert cube, I^∞, is the countably infinite product of unit intervals I. That is, $I^\infty = \prod\{I_i \mid i \in \mathbb{N}\}$ where $I_i = I$ for each i. The Hilbert cube has pairs (A_i, B_i) of **opposite faces**. These are defined by, $A_i = \{(x_j) \in I^\infty \mid x_i = 0\}$ and $B_i = \{(x_j) \in I^\infty \mid x_i = 1\}$.

Theorem 6.6. *The collection of opposite face pairs $\{(A_i, B_i) \mid i \in \mathbb{N}\}$ in the Hilbert cube is an essential family. Therefore the Hilbert cube is strongly infinite-dimensional.*

In I^∞, for each $i \in \mathbb{N}$, let $X_i = \{(x_j) \mid x_j = 0 \text{ for } j > i\}$. Put $X_\infty = \bigcup\{X_i \mid i \in \mathbb{N}\}$.

Proposition 6.7. *X_∞ is weakly infinite-dimensional.*

It is an interesting fact that there exist strongly infinite-dimensional metrizable compacta X having the property that if Y is a nonempty subspace of X, then either $\dim Y = 0$ or Y is strongly infinite-dimensional. It is even more surprising that one may obtain such X which is totally disconnected. We shall not pursue these topics here.

7. Cohomological Dimension

The definition of $\dim_G$ for any abelian group G was given in Section 2 making use of Čech cohomology. What is important for our needs is that for each topological space X, closed subspace A of X, abelian group G, and nonnegative integer n, there is a well-defined reduced Čech cohomology group, denoted $\tilde{\check{H}}^n(X, A; G)$. This group is always abelian. The theory agrees with singular cohomology when (X, A) is a polyhedral pair; we will discuss polyhedra in Section 9.

The case in which $G = \mathbb{Z}$ is special. Let us state Alexandroff's Theorem comparing dim and $\dim_{\mathbb{Z}}$.

Theorem 7.1. *Let X be a compact Hausdorff space with $\dim X < \infty$. Then $\dim_{\mathbb{Z}} X = \dim X$.*

As surprising as this theorem may seem, once it is accepted, the next result is even more surprising. It is due to A. Dranishnikov ([15]).

Theorem 7.2. *There exists a metrizable compactum X such that $\dim X = \infty$ and $\dim_{\mathbb{Z}} X < \infty$.*

As mentioned in Section 2, $\dim_G$ can be defined as $\mathscr{K}$-dim when $\mathscr{K} = \{K(G, i) \mid i \in \mathbb{N}\}$. What are the Eilenberg-MacLane CW-complexes $K(G, i)$? We shall assume that the reader is familiar with the idea of a CW-complex. Reference [55] can be consulted or one may find such information in [98].