Chapter 4

Converse inequalities in compact Hausdorff space

In this chapter we will prove difference type converses of the Jensen and Edmundson-Lah-Ribarič operator inequality for a unital field of positive linear mappings between C^*-algebras of operators in compact Hausdorff space, as well as further refinements and improvements thereto. Obtained general result will be applied to quasi-arithmetic operator means and to potential operator means with the aim of obtaining a better estimate of the difference between these means. Likewise, the mutual bounds for the Jensen operator inequality and the Lah-Ribarič operator inequality for the classes of bounded real-valued functions and Lipschitzian functions will be studied. The connection with the classical convexity will also be discussed. In the last section we give several mutual bounds for the operator version of the Edmundson-Lah-Ribarič inequality which hold for the class of n-convex functions. By virtue of the established estimates, we then derive several mutual bounds for the Jensen operator inequality which are also related to n-convex functions. As an application, we obtain mutual bounds for the differences of quasi-arithmetic and power operator means based on n-convexity.

4.1 Introduction

Let T be a locally compact Hausdorff space and let $\mathscr{A}$ be a C^*-algebra. We say that a field $(x_t)_{t\in T}$ of elements in $\mathscr{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. Additionally, if T is equipped with a Radon measure μ and the function $t \to ||x_t||$ is integrable, then, the so-called Bochner integral $\int_T x_t d\mu(t)$ can be formed. More precisely, the Bochner integral is the unique element in $\mathscr{A}$ such that the relation

$$\varphi\left(\int_T x_t d\mu(t)\right) = \int_T \varphi(x_t)\, d\mu(t)$$

holds for every linear functional φ in the norm dual $\mathscr{A}^*$ (see [55]).

Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ from $\mathscr{A}$ to another C^*-algebra $\mathscr{B}$. Such field is said to be continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathscr{A}$. If the C^*-algebras are unital and the field $t \to \phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital. We assume that such field is continuous.

Let x and y be operators (acting) on an infinite dimensional Hilbert space $\mathscr{H}$. The ordering is defined by setting $x \le y$ if $y - x$ is a positive semi-definite operator.

A continuous function $f : I \to \mathbb{R}$ is operator convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

holds for each $\lambda \in [0,1]$ and every pair of self-adjoint operators x and y (acting) on an infinite dimensional Hilbert space $\mathscr{H}$ with spectra in I. When the inequality sign is reversed, function f is operator concave.

If $f : I \to \mathbb{R}$ is operator convex function, where I is a real interval of any type, and $(\phi_t)_{t\in T}$ is a unital field, then the Jensen operator inequality (see Hansen *et.al.*, [56]) asserts that

$$f\left(\int_T \phi_t(x_t) d\mu(t)\right) \le \int_T \phi_t(f(x_t)) d\mu(t) \tag{4.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in I. If $f : I \to \mathbb{R}$ is operator concave function, then the sign of inequality in (4.1) is reversed.

In the same paper, Hansen *et.al.* obtained the following inequality which holds for an usual convex function $f : [m,M] \to \mathbb{R}$ (see [56], proof of Theorem 2):

$$\int_T \phi_t(f(x_t)) d\mu(t) \le \alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1}. \tag{4.2}$$

In this matter, the usual notation is used:

$$\alpha_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f = \frac{Mf(m) - mf(M)}{M - m}.$$

Inequality (4.2) is referred to as the Edmundson-Lah-Ribarič operator inequality. Observe that the operator inequality (4.2) is established by applying the functional calculus to the well-known inequality

$$f(t) \le \alpha_f t + \beta_t, \tag{4.3}$$

which holds for every convex function on the interval $[m,M]$. Recall that $l(t) = \alpha_f t + \beta_t$ is the linear function limiting convex function $f(t)$ on interval $[m,M]$ from the above.

On the other hand, Mićić, Pečarić and Perić in [99] obtained the following improvement of the Edmundson-Lah-Ribarič operator inequality

$$\int_T \phi_t(f(\mathbf{x}_t))d\mu(t) \le \alpha_f \int_T \phi_t(\mathbf{x}_t)d\mu(t) + \beta_f \mathbf{1} - \delta_f \underline{\mathbf{x}}, \tag{4.4}$$

where

$$\underline{\mathbf{x}} = \frac{1}{2}\mathbf{1} - \frac{1}{M-m}\int_T \phi_t\left(\left|\mathbf{x}_t - \frac{m+M}{2}\mathbf{1}\right|\right)d\mu(t) \tag{4.5}$$

and

$$\delta_f = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right). \tag{4.6}$$

Since $f : [m,M] \to \mathbb{R}$ is convex function, it follows that $\underline{\mathbf{x}} \ge \mathbf{0}$ and $\delta_f \ge 0$.

The techniques that will be used in the proofs are mainly based on the classical real and functional calculus, especially on the well-known monotonicity principle for self-adjoint elements of a C^*-algebra $\mathscr{A}$: If $\mathbf{x} \in \mathscr{A}$ with a spectra $\mathrm{Sp}(\mathbf{x})$, then

$$f(t) \ge g(t),\ t \in \mathrm{Sp}(x) \quad \Longrightarrow \quad f(x) \ge g(x), \tag{4.7}$$

where f and g are real continuous functions (for more details see [49]). Moreover, all the results that follow include the Bochner integral, defined in this Introduction. If nothing else is explicitly stated, $(\mathbf{x}_t)_{t\in T}$ is a bounded continuous field of self-adjoint elements in unital C^*-algebra whose spectra belongs to a domain of the corresponding function and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings between the corresponding unital C^*-algebras.

The following results refer to functions that are convex in the classical sense. Although regarding different inequalities, it appears that these two series of converses are closely connected.

4.2 Converses of the Jensen and Edmundson-Lah--Ribarič operator inequality

First we give a series of converses for the Jensen operator inequality obtained in [65]. It should be noticed here that the following theorem in the classical real case was proved by Dragomir in the recent paper [37], and generalization of the same inequality for linear functionals was proved by Jakšić and Pečarić in [68]. In fact, such series of scalar inequalities will be exploited in establishing the corresponding operator form.

Theorem 4.1 *Let $f : I \to \mathbb{R}$ be a continuous convex function, and let $m, M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned} \tag{4.8}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.8) *are reversed.*

Proof. Taking into account the operator version of the Lah-Ribarič inequality (4.2), it follows that

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\leq \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right).
\end{aligned} \tag{4.9}$$

On the other hand, regarding convexity of f, we have the so-called gradient inequality,

$$f(t) - f(M) \geq f'_-(M)(t-M),$$

which holds for every $t \in [m,M]$, that is,

$$(t-m)f(t) - (t-m)f(M) \geq f'_-(M)(t-M)(t-m), \quad t \in [m,M],$$

after multiplying with $t-m$. In the same way, it follows that

$$(M-t)f(t) - (M-t)f(m) \geq f'_+(m)(M-t)(t-m), \quad t \in [m,M].$$

Now, adding the above two inequalities, and then, dividing by $m-M$, we have

$$\alpha_f t + \beta_f - f(t) \leq \frac{f'_-(M) - f'_+(m)}{M-m}(M-t)(t-m). \tag{4.10}$$

Moreover, taking into account the arithmetic-geometric mean inequality, the following series of inequalities holds for all $t \in [m,M]$ (see also [37]):

$$\begin{aligned}
\alpha_f t + \beta_f - f(t) &\leq \frac{f'_-(M) - f'_+(m)}{M-m}(M-t)(t-m) \\
&\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)).
\end{aligned} \tag{4.11}$$

Now, since $m\mathbf{1} \le x_t \le M\mathbf{1}$ for every $t \in T$, it follows that $m\phi_t(\mathbf{1}) \le \phi_t(x_t) \le M\phi_t(\mathbf{1})$, that is, $m\mathbf{1} \le \int_T \phi_t(x_t)d\mu(t) \le M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, setting $\int_T \phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}
&\alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right) \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}. \qquad (4.12)
\end{aligned}$$

Finally, comparing (4.9) and (4.12), we obtain (4.8), as claimed. □

Remark 4.1 Observe that in the statement of Theorem 4.1 the interval $[m,M]$ belongs to the interior of the interval I. This condition assures finiteness of the one-sided derivatives in (4.8). Without this assumption these derivatives might be infinite.

Remark 4.2 It should be noticed here that the first expression in the series of inequalities (4.8), that is, the element $\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (4.1).

The following result was also proved in [65] and it represents converses of the Edmundson-Lah-Ribarič operator inequality (4.2):

Theorem 4.2 *Suppose $f : I \to \mathbb{R}$ is a continuous convex function, and $m, M \in \mathbb{R}$, $m < M$, are such that interval $[m,M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
0 &\le \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m}\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
&\le \frac{f'_-(M) - f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
&\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1} \qquad (4.13)
\end{aligned}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.13) are reversed.

Proof. The first inequality in (4.13) holds by virtue of the Edmundson-Lah-Ribarič inequality (4.2). Further, starting from the scalar inequality (4.10), it follows that relation

$$\alpha_f x_t + \beta_f \mathbf{1} - f(x_t) \le \frac{f'_-(M) - f'_+(m)}{M-m}(M\mathbf{1} - x_t)(x_t - m\mathbf{1})$$

holds for every $t \in T$. Now, applying the positive linear mappings ϕ_t to the above relation, we obtain

$$\alpha_f \phi_t(x_t) + \beta_f \phi_t(\mathbf{1}) - \phi_t(f(x_t)) \leq \frac{f'_-(M) - f'_+(m)}{M-m} \phi_t([M\mathbf{1} - x_t][x_t - m\mathbf{1}]),$$

while integrating yields

$$\begin{aligned} &\alpha_f \int_T \phi_t(x_t) d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t)) d\mu(t) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t), \end{aligned}$$

so the second inequality in (4.13) holds.

Taking into account Theorem 4.1, it is enough to justify the third inequality sign in (4.13). To prove our assertion, we note that the function

$$h(t) = (M-t)(t-m) = -t^2 + (M+m)t - Mm, \quad t \in [m, M]$$

is operator concave (see e.g. [49]). Finally, applying the Jensen operator inequality (4.1) to the above function h, it follows that

$$\begin{aligned} &\int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right) d\mu(t) \\ &\leq \left(M\mathbf{1} - \int_T \phi_t(x_t) d\mu(t)\right)\left(\int_T \phi_t(x_t) d\mu(t) - m\mathbf{1}\right), \end{aligned}$$

and the proof is completed. □

Following results are obtained in [66] and they represent more precise converses of the Jensen and Edmundson-Lah-Ribarič operator inequality, and they represent also refinements of the inequalities (4.8) and (4.13) respectively. Such improved relations are also accompanied with a convexity in the classical real sense.

In order to present our basic results, we define

$$\Delta_f(t; m, M) = \frac{1}{M-m}\left[\frac{f(M) - f(t)}{M-t} - \frac{f(t) - f(m)}{t-m}\right], \tag{4.14}$$

where $m < M$ and $f : I \to \mathbb{R}$ is a continuous convex function such that the interval $[m, M]$ belongs to the interior of interval I. Observe that expression (4.14) is actually the second order divided difference of the function f at points m, t, and M, for every $t \in (m, M)$.

Remark 4.3 Observe that the function f is defined on the interval I whose interior contains interval $[m, M]$. This condition ensures finiteness of one-sided derivatives at points m and M. Then,

$$\lim_{t \to m^+} \Delta_f(t; m, M) = \frac{1}{M-m}\left[\frac{f(M) - f(m)}{M-m} - f'_+(m)\right]$$

and

$$\lim_{t\to M^-} \Delta_f(t;m,M) = \frac{1}{M-m}\left[f'_-(M) - \frac{f(M)-f(m)}{M-m}\right],$$

so $\Delta_f(\cdot;m,M)$ may be regarded as a continuous function (in parameter t) on the interval $[m,M]$. Therefore, if x is a self-adjoint element in C^*-algebra with spectra contained in $[m,M]$, then the expression $\Delta_f(x;m,M)$ is also meaningful. Clearly, this assertion holds due to functional calculus.

Now we give two series of converses for the Jensen operator inequality. One of them refines series (4.8). The classical real version of the following theorem was proved by Dragomir in recent paper [38]. In fact, such scalar series of inequalities will be exploited in establishing the corresponding operator forms.

Theorem 4.3 *Let $f: I \to \mathbb{R}$ be a continuous convex function, and let $m,M \in \mathbb{R}$, $m < M$, be such that interval $[m,M]$ belongs to the interior of interval I. Further, suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, and $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t: \mathscr{A} \to \mathscr{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\le \sup_{m<t<M} \Delta_f(t;m,M)\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)\\
&\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right)\\
&\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\qquad\le \frac{1}{4}(M-m)^2\Delta_f\left(\int_T \phi_t(x_t)d\mu(t);m,M\right)\\
&\qquad\le \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}
\end{aligned} \tag{4.16}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m,M]$. If f is concave on I, then the signs of inequalities in (4.15) *and* (4.16) *are reversed.*

Proof. Taking into account the operator version of the Edmundson-Lah-Ribarič inequality (4.2), it follows that

$$\begin{aligned}
&\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)\\
&\le \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f\mathbf{1} - f\left(\int_T \phi_t(x_t)d\mu(t)\right).
\end{aligned} \tag{4.17}$$

On the other hand, the scalar inequality

$$\begin{aligned}\alpha_f t+\beta_f - f(t) &= \frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)-f(t)\\ &= \frac{(M-t)(t-m)}{M-m}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right]\\ &= (M-t)(t-m)\Delta_f(t;m,M)\\ &\leq (M-t)(t-m)\sup_{m<t<M}\Delta_f(t;m,M) \qquad (4.18)\end{aligned}$$

holds for all $t\in[m,M]$. In addition, since

$$\begin{aligned}\sup_{m<t<M}\Delta_f(t;m,M) &= \frac{1}{M-m}\sup_{m<t<M}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right]\\ &\leq \frac{1}{M-m}\left[\sup_{m<t<M}\frac{f(M)-f(t)}{M-t}+\sup_{m<t<M}\left(-\frac{f(t)-f(m)}{t-m}\right)\right]\\ &= \frac{1}{M-m}\left[\sup_{m<t<M}\frac{f(M)-f(t)}{M-t}-\inf_{m<t<M}\frac{f(t)-f(m)}{t-m}\right]\\ &= \frac{f'_-(M)-f'_+(m)}{M-m}, \qquad (4.19)\end{aligned}$$

we have the following series of inequalities:

$$\begin{aligned}\alpha_f t+\beta_f-f(t) &\leq (M-t)(t-m)\sup_{m<t<M}\Delta_f(t;m,M)\\ &\leq \frac{f'_-(M)-f'_+(m)}{M-m}(M-t)(t-m)\\ &\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m)). \qquad (4.20)\end{aligned}$$

Clearly, the last inequality sign in (4.20) holds due to the arithmetic-geometric mean inequality, that is, $(M-t)(t-m)\leq\frac{1}{4}(M-m)^2$.

Now, since $m\mathbf{1}\leq x_t\leq M\mathbf{1}$ for every $t\in T$, it follows that $m\phi_t(\mathbf{1})\leq\phi_t(x_t)\leq M\phi_t(\mathbf{1})$, that is, $m\mathbf{1}\leq\int_T\phi_t(x_t)d\mu(t)\leq M\mathbf{1}$. Hence, applying the functional calculus to the above series of inequalities, that is, putting $\int_T\phi_t(x_t)d\mu(t)$ instead of t, we have

$$\begin{aligned}&\alpha_f\int_T\phi_t(x_t)d\mu(t)+\beta_f\mathbf{1}-f\left(\int_T\phi_t(x_t)d\mu(t)\right)\\ &\leq \sup_{m<t<M}\Delta_f(t;m,M)\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\ &\leq \frac{f'_-(M)-f'_+(m)}{M-m}\left(M\mathbf{1}-\int_T\phi_t(x_t)d\mu(t)\right)\left(\int_T\phi_t(x_t)d\mu(t)-m\mathbf{1}\right)\\ &\leq \frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\mathbf{1}. \qquad (4.21)\end{aligned}$$

Finally, comparing (4.17) and (4.21), we obtain (4.15), as claimed.

To prove (4.16), we start with the scalar series of inequalities

$$\begin{aligned}\alpha_f t + \beta_f - f(t) &\le \frac{1}{4}(M-m)^2 \Delta_f(t;m,M) \\ &\le \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)),\ t \in [m,M], \end{aligned} \tag{4.22}$$

which obviously follows from (4.18), (4.19), and the arithmetic-geometric mean inequality. Finally, setting $\int_T \phi_t(x_t)d\mu(t)$ in (4.22) and utilizing (4.17), we obtain (4.16) and the proof is completed. □

Remark 4.4 Observe that the series of inequalities in (4.15) refines the series (4.8), since $\sup_{m<t<M} \Delta_f(t;m,M) \le \frac{f'_-(M)-f'_+(m)}{M-m}$. For example, if $f(t) = t^2$ and $m < M$, then

$$1 = \sup_{m<t<M} \Delta_f(t;m,M) < \frac{f'_-(M) - f'_+(m)}{M-m} = 2,$$

while for $f(t) = t^3$ we have

$$m + 2M = \sup_{m<t<M} \Delta_f(t;m,M) < \frac{f'_-(M) - f'_+(m)}{M-m} = 3(m+M),$$

provided that $0 < m < M$. However, a convex function needs not to be differentiable. To see the corresponding example, let $m < 0 < M$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} t^2, & t \ge 0 \\ -t, & t < 0 \end{cases}.$$

Then,

$$\Delta_f(t;m,M) = \begin{cases} 1 - \frac{m(m+1)}{(M-m)(t-m)}, & t \ge 0 \\ \frac{M(M+1)}{(M-m)(M-t)}, & t < 0 \end{cases},$$

and consequently,

$$\sup_{m<t<M} \Delta_f(t;m,M) = \begin{cases} \frac{M^2-2Mm-m}{(M-m)^2}, & \text{if } m < -1 \\ \frac{M+1}{M-m}, & \text{if } -1 \le m < 0 \end{cases}.$$

On the other hand,

$$\frac{f'_-(M) - f'_+(m)}{M-m} = \frac{2M+1}{M-m},$$

which implies that $\sup_{m<t<M} \Delta_f(t;m,M) < \frac{f'_-(M)-f'_+(m)}{M-m}$, since $M > 0$.

Remark 4.5 It should be noticed here that the first line in the series of inequalities (4.15) and (4.16), that is, the element $\int_T \phi_t(f(x_t))d\mu(t) - f\left(\int_T \phi_t(x_t)d\mu(t)\right)$ is not positive in general. This element is positive if f is in addition operator convex function, due to the Jensen operator inequality (4.1).

The following result provides several converse series of inequalities for the Edmundson-Lah-Ribarič operator inequality (4.2). As we shall see below, one of them improves the series (4.13).

Theorem 4.4 *Suppose $f : I \to \mathbb{R}$ is a continuous convex function, and $m, M \in \mathbb{R}$, $m < M$, are such that interval $[m, M]$ belongs to the interior of interval I. Further, let $(\phi_t)_{t\in T}$ be a unital field of positive linear mappings $\phi_t : \mathscr{A} \to \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are unital C^*-algebras, defined on a locally compact Hausdorff space T with a bounded Radon measure μ. Then the series of inequalities*

$$\begin{aligned}
0 \leq& \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq& \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
\leq& \frac{f'_-(M) - f'_+(m)}{M-m} \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
\leq& \frac{f'_-(M) - f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq& \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
0 \leq& \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq& \sup_{m<t<M} \Delta_f(t;m,M) \int_T \phi_t\left([M\mathbf{1} - x_t][x_t - m\mathbf{1}]\right)d\mu(t) \\
\leq& \sup_{m<t<M} \Delta_f(t;m,M) \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq& \frac{f'_-(M) - f'_+(m)}{M-m} \left(M\mathbf{1} - \int_T \phi_t(x_t)d\mu(t)\right)\left(\int_T \phi_t(x_t)d\mu(t) - m\mathbf{1}\right) \\
\leq& \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1},
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
0 \leq& \alpha_f \int_T \phi_t(x_t)d\mu(t) + \beta_f \mathbf{1} - \int_T \phi_t(f(x_t))d\mu(t) \\
\leq& \frac{1}{4}(M-m)^2 \int_T \phi_t\left(\Delta_f(x_t;m,M)\right)d\mu(t) \\
\leq& \frac{1}{4}(M-m)(f'_-(M) - f'_+(m))\mathbf{1}
\end{aligned} \tag{4.25}$$

hold for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathscr{A}$ with spectra contained in $[m, M]$. Moreover, if f is concave on I, then the signs of inequalities in (4.23)*,* (4.24)*, and* (4.25) *are reversed.*