Chapter 3

Class of $(h, g; m)$-convex functions and certain types of inequalities

A convex function is one whose epigraph is a convex set, or, as in the basic definition:

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{3.1}$$

holds for all points x and y in I and all $\lambda \in [0,1]$.

It is called strictly convex if the inequality (3.1) *holds strictly whenever x and y are distinct points and $\lambda \in (0,1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.*

Motivated by a large number of different classes of convexity, we present a new convexity that unifies a certain range of them. Starting from the above convex function up to a recent convexity [27]:

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called exponentially (s,m)-convex in the second sense if the following inequality holds

$$f(\lambda x + m(1-\lambda) y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} m f(y) \tag{3.2}$$

for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$, $s, m \in (0,1]$.

we noticed that the whole range in-between could be covered if we use on the right-hand side functions h and g in a form

$$f(\lambda x+m(1-\lambda)y)\le h(\lambda)f(x)g(x)+mh(1-\lambda)f(y)g(y).$$

We named this convexity an $(h,g;m)$-convexity.

Here are several more varieties of convexity that will be generalized with this:

- A non-negative function $f: I\subset\mathbb{R}\to\mathbb{R}$ is called P-function if the inequality holds
$$f(\lambda x+(1-\lambda)y)\le f(x)+f(y)$$
for all $x,y\in I$ and all $\lambda\in[0,1]$.
- A function $f:[0,\infty)\to[0,\infty)$ is called s-convex in the second sense if the inequality holds
$$f(\lambda x+(1-\lambda)y)\le\lambda^s f(x)+(1-\lambda)^s f(y)$$
for all $x,y\in[0,\infty)$ and all $\lambda\in[0,1]$, where $s\in(0,1]$.
- A non-negative function $f: I\subset\mathbb{R}\to\mathbb{R}$ is called Godunova-Levin function if the inequality holds
$$f(\lambda x+(1-\lambda)y)\le\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}$$
for all $x,y\in I$ and all $\lambda\in(0,1)$.
- A non-negative function $f: I\subset\mathbb{R}\to\mathbb{R}$ is called h-convex if the inequality holds
$$f(\lambda x+(1-\lambda)y)\le h(\lambda)f(x)+h(1-\lambda)f(y)$$
for all $x,y\in I$ and all $\lambda\in(0,1)$, where $h: J\to\mathbb{R}$ is a non-negative function, $h\not\equiv 0$, $(0,1)\subseteq J$.
- A function $f:[0,b]\to\mathbb{R}$ is called m-convex if the inequality holds
$$f(\lambda x+m(1-\lambda)y)\le\lambda f(x)+m(1-\lambda)f(y)$$
for all $x,y\in[0,b]$ and all $\lambda\in[0,1]$, where $m\in[0,1]$.
- A non-negative function $f:[0,b]\to\mathbb{R}$ is called $(h-m)$-convex if the inequality holds
$$f(\lambda x+m(1-\lambda)y)\le h(\lambda)f(x)+mh(1-\lambda)f(y)$$
for all $x,y\in[0,b]$ and all $\lambda\in(0,1)$, where $h: J\to\mathbb{R}$ is a non-negative function, $h\not\equiv 0$, $(0,1)\subseteq J$ and $m\in[0,1]$.
- A non-negative function $f: I\subset\mathbb{R}\to\mathbb{R}$ is called (s,m)-Godunova-Levin function of the second kind if the inequality holds
$$f(\lambda x+m(1-\lambda)y)\le\frac{f(x)}{\lambda^s}+\frac{mf(y)}{(1-\lambda)^s}$$
for all $x,y\in I$ and all $\lambda\in(0,1)$, where $m\in(0,1]$, $s\in[0,1]$.

- A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called exponential convex if the inequality holds
$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda}{e^{\alpha x}} f(x) + \frac{1-\lambda}{e^{\alpha y}} f(y)$$
for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$.

- A function $f: I \subset \mathbb{R} \to \mathbb{R}$ is called exponentially s-convex in the second sense if the inequality holds
$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} f(y)$$
for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$, $s \in (0,1]$.

More detailed information may be found in [8, 10, 12, 15, 20, 23, 24, 27, 35, 36].

Furthermore, recall that a real valued function f on the interval I is said to be starshaped if
$$f(\lambda x) \leq \lambda f(x)$$
whenever $x \in I, \lambda x \in I$ and $\lambda \in [0,1]$.

This chapter is based on our results from [1], [2], [3], [6] and [7].

3.1 A class of $(h,g;m)$-convex functions

Definition 3.1 *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and let g be a positive function on $I \subseteq \mathbb{R}$. Furthermore, let $m \in (0,1]$. A function $f: I \to \mathbb{R}$ is said to be an $(h,g;m)$-convex function if it is nonnegative and if*
$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y) \tag{3.3}$$
holds for all $x, y \in I$ and all $\lambda \in (0,1)$.

If (3.3) *holds in the reversed sense, then f is said to be an $(h,g;m)$-concave function.*

Remark 3.1 For different choices of functions h, g and parameter m in (3.3), we can obtain corresponding convexity, e.g., if we set $h(\lambda) = \lambda^s$, $s \in (0,1]$, $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, then $(h,g;m)$-convexity reduces to exponentially (s,m)-convexity in the second sense (3.2).

Lemma 3.1 *If $f: I \to [0,\infty)$ is an $(h,g;m)$-convex function such that $f(0) = 0$, $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then f is starshaped.*

Proof. Let f be an $(h,g;m)$-convex function. Then we have
$$\begin{aligned} f(\lambda x) &= f(\lambda x + m(1-\lambda)0) \\ &\leq h(\lambda)f(x)g(x) + mh(1-\lambda)f(0)g(0) \\ &\leq \lambda f(x). \end{aligned}$$
Therefore, f is a starshaped. □

Remark 3.2 Let g be a positive function such that $g(x) \geq 1$. If f is a nonnegative $(h-m)$-convex function on $[0,\infty)$, then we have

$$\begin{aligned} f(\lambda x+m(1-\lambda)y) &\leq h(\lambda)f(x)+mh(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x)+mh(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h,g;m)$-convex function.

If additionally $h(\lambda) \geq \lambda$, then for nonnegative m-convex function f on $[0,\infty)$ we have

$$\begin{aligned} f(\lambda x+m(1-\lambda)y) &\leq \lambda f(x)+m(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)+mh(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x)+mh(1-\lambda)f(y)g(y), \end{aligned}$$

i.e., f is an $(h,g;m)$-convex function. An example of a function that satisfies $h(\lambda) \geq \lambda$ is $h(\lambda) = \lambda^k$, where $k \leq 1$ and $\lambda \in (0,1)$.

Similarly, if $g(x) \leq 1$, then all nonnegative $(h-m)$-concave functions are $(h,g;m)$-concave functions on $[0,\infty)$. Furthermore, if $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then all nonnegative m-concave functions are $(h,g;m)$-concave functions on $[0,\infty)$.

Proposition 3.1 *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$, such that*

$$h_2(\lambda) \leq h_1(\lambda), \quad \lambda \in (0,1).$$

Let g be a positive function on $I \subseteq \mathbb{R}$ and $m \in (0,1]$. If $f : I \to [0,\infty)$ is an $(h_2,g;m)$-convex function, then f is $(h_1,g;m)$-convex.

If $f : I \to [0,\infty)$ is an $(h_1,g;m)$-concave function, then f is $(h_2,g;m)$-concave.

Proof. Let f be an $(h_2,g;m)$-convex function. Then we have

$$\begin{aligned} f(\lambda x+m(1-\lambda)y) &\leq h_2(\lambda)f(x)g(x)+mh_2(1-\lambda)f(y)g(y) \\ &\leq h_1(\lambda)f(x)g(x)+mh_1(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h_1,g;m)$-convex function.

If f is an $(h_1,g;m)$-concave function, then analogously follows that f is $(h_2,g;m)$-concave. □

Proposition 3.2 *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Furthermore, let $m \in (0,1]$ and $\alpha > 0$. If $f_1, f_2 : I \to [0,\infty)$ are $(h,g;m)$-convex functions, then $f_1 + f_2$ and αf_1 are $(h,g;m)$-convex.*

If $f_1, f_2 : I \to [0,\infty)$ are $(h,g;m)$-concave functions, then $f_1 + f_2$ and αf_1 are $(h,g;m)$-concave.

Proof. Let f_1, f_2 be $(h,g;m)$-convex functions and $\alpha > 0$. Then we have

$$f_1(\lambda x+m(1-\lambda)y) \leq h(\lambda)f_1(x)g(x)+mh(1-\lambda)f_1(y)g(y)$$

and

$$f_2(\lambda x+m(1-\lambda)y) \le h(\lambda)f_2(x)g(x)+mh(1-\lambda)f_2(y)g(y).$$

Adding the above we obtain

$$[f_1+f_2](\lambda x+m(1-\lambda)y) \le h(\lambda)[f_1+f_2](x)g(x)+mh(1-\lambda)[f_1+f_2](y)g(y).$$

Furthermore,

$$\begin{aligned}[\alpha f_1](\lambda x+m(1-\lambda)y) &\le \alpha h(\lambda)f_1(x)g(x)+\alpha mh(1-\lambda)f_1(y)g(y)\\ &= h(\lambda)[\alpha f_1](x)g(x)+mh(1-\lambda)[\alpha f_1](y)g(y).\end{aligned}$$

We conclude that f_1+f_2 and αf_1 are $(h_1,g;m)$-convex.

If $f_1,f_2 : I\to[0,\infty)$ are $(h,g;m)$-concave functions, then analogously follows that f_1+f_2 and αf_1 are $(h,g;m)$-concave. □

Proposition 3.3 *Let h be a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$ and g be a positive increasing function on $I\subseteq\mathbb{R}$. Furthermore, let $0<n<m\le 1$. If $f : I\to[0,\infty)$ is an $(h,g;m)$-convex function such that $f(0)=0$, $g(x)\le 1$ and $h(\lambda)\le\lambda$, then f is $(h,g;n)$-convex.*

Proof. Let f be an $(h,g;m)$-convex function. From $f(0)=0$, $g(x)\le 1$ and $h(\lambda)\le\lambda$ by Lemma 3.1 follows $f(\lambda x)\le\lambda f(x)$. Considering also that g is an increasing function, we obtain

$$\begin{aligned}f(\lambda x+n(1-\lambda)y) &= f\left(\lambda x+m(1-\lambda)\left(\frac{n}{m}y\right)\right)\\ &\le h(\lambda)f(x)g(x)+mh(1-\lambda)f\left(\frac{n}{m}y\right)g\left(\frac{n}{m}y\right)\\ &\le h(\lambda)f(x)g(x)+mh(1-\lambda)\frac{n}{m}f(y)g(y),\end{aligned}$$

which proves that f is $(h,g;n)$-convex. □

Proposition 3.4 *Let h_1,h_2 be nonnegative functions on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h_1,h_2\not\equiv 0$ and let*

$$h(t)=\max\{h_1(t),h_2(t)\},\quad t\in J.$$

Let g_1,g_2 be positive functions on $I\subseteq\mathbb{R}$ and let $m_1,m_2\in(0,1]$. For $i=1,2$, let $f_i : I\to[0,\infty)$ be $(h_i,g_i;m_i)$-convex functions. If the functions f_1g_1 and f_2g_2 are monotonic in the same sense, i.e.

$$[f_1(x)g_1(x)-f_1(y)g_1(y)][f_2(x)g_2(x)-f_2(y)g_2(y)]\ge 0,\quad x,y\in I,$$

and if $c>0$ such that

$$h(\lambda)+mh(1-\lambda)\le c,\quad \lambda\in(0,1),$$

where $m=\max\{m_1,m_2\}$, then f_1f_2 is a $(ch,g_1g_2;m)$-convex function.

Proof. Let $f_i : I \to [0,\infty)$ be $(h_i,g_i;m_i)$-convex functions, $i = 1,2$. From hypotheses on functions, for $x,y \in I$ we have

$$\begin{aligned} & f_1(x)g_1(x)f_2(x)g_2(x) + f_1(y)g_1(y)f_2(y)g_2(y) \\ & \quad \geq f_1(x)g_1(x)f_2(y)g_2(y) + f_1(y)g_1(y)f_2(x)g_2(x). \end{aligned}$$

Let α and $\beta > 0$ be positive numbers such that $\alpha+\beta = 1$. Then we have

$$\begin{aligned} & f_1f_2(\alpha x+\beta y) \\ & \quad \leq [h_1(\alpha)f_1(x)g_1(x)+m_1h_1(\beta)f_1(y)g_1(y)] \\ & \qquad \times[h_2(\alpha)f_2(x)g_2(x)+m_2h_2(\beta)f_2(y)g_2(y)] \\ & \quad \leq [h(\alpha)f_1(x)g_1(x)+mh(\beta)f_1(y)g_1(y)] \\ & \qquad \times[h(\alpha)f_2(x)g_2(x)+mh(\beta)f_2(y)g_2(y)] \\ & \quad = h^2(\alpha)f_1(x)g_1(x)f_2(x)g_2(x)+mh(\alpha)h(\beta)f_1(x)g_1(x)f_2(y)g_2(y) \\ & \qquad +mh(\alpha)h(\beta)f_1(y)g_1(y)f_2(x)g_2(x)+m^2h^2(\beta)f_1(y)g_1(y)f_2(y)g_2(y), \end{aligned}$$

hence

$$\begin{aligned} & f_1f_2(\alpha x+\beta y) \\ & \quad \leq h^2(\alpha)f_1(x)g_1(x)f_2(x)g_2(x)+mh(\alpha)h(\beta)f_1(x)g_1(x)f_2(x)g_2(x) \\ & \qquad +mh(\alpha)h(\beta)f_1(y)g_1(y)f_2(y)g_2(y)+m^2h^2(\beta)f_1(y)g_1(y)f_2(y)g_2(y) \\ & \quad = [h(\alpha)+mh(\beta)] \\ & \qquad \times[h(\alpha)f_1(x)f_2(x)g_1(x)g_2(x)+mh(\beta)f_1(y)f_2(y)g_1(y)g_2(y)] \\ & \quad \leq ch(\alpha)f_1(x)f_2(x)g_1(x)g_2(x)+mch(\beta)f_1(y)f_2(y)g_1(y)g_2(y). \end{aligned}$$

This proves that $f_1 f_2$ is $(ch,g_1g_2;m)$-convex. □

Analogously follows the following proposition.

Proposition 3.5 *Let h_1,h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1,h_2 \not\equiv 0$ and let*

$$h(t) = \min\{h_1(t),h_2(t)\}, \quad t \in J.$$

Let g_1,g_2 be positive functions on $I \subseteq \mathbb{R}$ and let $m_1,m_2 \in (0,1]$. For $i = 1,2$, let $f_i : I \to [0,\infty)$ be $(h_i,g_i;m_i)$-concave functions. If the functions $f_1 g_1$ and $f_2 g_2$ are monotonic in the opposite sense, i.e.

$$[f_1(x)g_1(x)-f_1(y)g_1(y)]\,[f_2(x)g_2(x)-f_2(y)g_2(y)] \leq 0, \quad x,y \in I,$$

and if $c > 0$ such that

$$h(\lambda)+mh(1-\lambda) \geq c, \quad \lambda \in (0,1),$$

where $m = \min\{m_1,m_2\}$, then $f_1 f_2$ is a $(ch,g_1g_2;m)$-concave function.

3.2 Hermite-Hadamard type inequalities for $(h,g;m)$-convex functions

The famous Hermite-Hadamard inequality gives us an estimate of the (integral) mean value of a continuous convex function.

Theorem 3.1 (THE HERMITE-HADAMARD INEQUALITY) *Let $f:[a,b]\to\mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}.$$

Of course, equality holds in either side only for affine functions. In this section we prove the Hermite-Hadamard inequality for $(h,g;m)$-convex functions and we point out some special results. Furthermore, several known inequalities are improved.

Recall, by $L_p[a,b]$, $1\leq p<\infty$, the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$ is denoted.

Theorem 3.2 *Let f be a nonnegative $(h,g;m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$, g is a positive function on $[0,\infty)$ and $m\in(0,1]$. If $f,g,h\in L_1[a,b]$, where $0\leq a<b<\infty$, then the following inequalities hold*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)g(x)+mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right]dx\\
&\leq \frac{h\left(\frac{1}{2}\right)f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g(x)\,dx\\
&\quad+\frac{mh\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g(x)\,dx\\
&\quad+\frac{mh\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g\left(\frac{x}{m}\right)dx\\
&\quad+\frac{m^2h\left(\frac{1}{2}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g\left(\frac{x}{m}\right)dx.
\end{aligned}\tag{3.4}$$

Proof. Let f be an $(h,g;m)$-convex function. Then for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{x+my}{2}\right)\leq h\left(\frac{1}{2}\right)\left[f(x)g(x)+mf(y)g(y)\right].$$

Choosing $y\equiv\frac{y}{m}$ we obtain

$$f\left(\frac{x+y}{2}\right)\leq h\left(\frac{1}{2}\right)\left[f(x)g(x)+mf\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right)\right].\tag{3.5}$$

Let $x=\lambda a+(1-\lambda)b$ and $y=(1-\lambda)a+\lambda b$. Then

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)\Big[f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b) +mf\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\Big].$$

In the following step we will need to integrate the above over $\lambda\in[0,1]$. From

$$\int_0^1 f(\lambda a+(1-\lambda)b)g(\lambda a+(1-\lambda)b)\,d\lambda=\frac{1}{b-a}\int_a^b f(u)g(u)\,du$$

and

$$\int_0^1 f\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda=\frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du$$

we obtain

$$f\left(\frac{a+b}{2}\right)\leq\frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(u)g(u)+mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du. \tag{3.6}$$

By $(h,g;m)$-convexity of f we have

$$f(\lambda a+(1-\lambda)b)\leq h(\lambda)f(a)g(a)+mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

Multiplying the above inequality by $g(\lambda a+(1-\lambda)b)$ and integrating over $\lambda\in[0,1]$ we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)g(u)du &\leq f(a)g(a)\int_0^1 h(\lambda)g(\lambda a+(1-\lambda)b)\,d\lambda\\ &\quad+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-\lambda)g(\lambda a+(1-\lambda)b)\,d\lambda\\ &=\frac{f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g(u)\,du\\ &\quad+\frac{mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g(u)\,du.\end{aligned} \tag{3.7}$$

Again, by $(h,g;m)$-convexity of f we have

$$f\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)\leq h(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+mh(\lambda)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)$$

and if we multiply above inequality by $g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)$ and integrate over $\lambda\in[0,1]$ we obtain

$$\begin{aligned}\frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du \le\; & f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-\lambda)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda \\ & +mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_0^1 h(\lambda)g\left((1-\lambda)\frac{a}{m}+\lambda\frac{b}{m}\right)d\lambda \\ =\; & \frac{f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g\left(\frac{u}{m}\right)du \\ & +\frac{mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g\left(\frac{u}{m}\right)du. \qquad (3.8)\end{aligned}$$

Now from (3.6), (3.7) and (3.8) we obtain (3.4). □

In the sequel we state several corollaries, using special functions for h and/or g, and choosing the parameter m. We start with the first special case: if $g\equiv 1$, then we have the Hermite-Hadamard inequality for $(h-m)$-convex functions.

Corollary 3.1 *Let f be a nonnegative $(h-m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$ and $m\in(0,1]$. If $f,h\in L_1[a,b]$, where $0\le a<b<\infty$, then the following inequalities hold*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \\ &\le h\left(\frac{1}{2}\right)\int_0^1 h(x)dx\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)\right]. \qquad (3.9)\end{aligned}$$

Proof. We use

$$\int_a^b h\left(\frac{b-x}{b-a}\right)dx=\int_a^b h\left(\frac{x-a}{b-a}\right)dx=(b-a)\int_0^1 h(u)\,du.$$

□

Remark 3.3 In [24, Theorem 9] authors gave the following Hermite-Hadamard type inequality for $(h-m)$-convex functions:

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \\ &\le h\left(\frac{1}{2}\right)\left[f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)\right]. \qquad (3.10)\end{aligned}$$

For all functions h such that $\int_0^1 h(x)\,dx\le 1$, our result (3.9) will improve (3.10).

If $g\equiv 1$ and $m=1$, then we have the Hermite-Hadamard inequality for h-convex functions ([36]):

Corollary 3.2 *Let f be a nonnegative h-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$. If $f, h \in L_1[a,b]$, where $0 \leq a < b < \infty$, then the following inequalities hold*

$$\frac{1}{2} f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a} \int_a^b f(x)\,dx \leq h\left(\frac{1}{2}\right) [f(a)+f(b)] \int_0^1 h(x)\,dx. \tag{3.11}$$

For h being identity and $g \equiv 1$, the Hermite-Hadamard type inequality for m-convex functions holds ([11]):

Corollary 3.3 *Let f be a nonnegative m-convex function on $[0,\infty)$ with $m \in (0,1]$. If $f \in L_1[a,b]$, where $0 \leq a < b < \infty$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx \leq \frac{1}{4}\left[f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right].$$

Of course, if $h(x) = x$, $g \equiv 1$ and $m = 1$, then we have the Hermite-Hadamard inequality given in Theorem 3.1.

An interesting Hermite-Hadamard type inequality follows if h is an identity.

Corollary 3.4 *Suppose that assumptions of Theorem 3.2 hold and let $h(x) = x$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)} \int_a^b \left[f(x)g(x) + mf\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right)\right] dx \\ &\leq \frac{f(a)g(a)}{2(b-a)^2} \int_a^b (b-x)g(x)\,dx \\ &\quad + \frac{mf\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)}{2(b-a)^2} \int_a^b (x-a)g(x)\,dx \\ &\quad + \frac{mf\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right)}{2(b-a)^2} \int_a^b (b-x)g\left(\frac{x}{m}\right) dx \\ &\quad + \frac{m^2 f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right)}{2(b-a)^2} \int_a^b (x-a)g\left(\frac{x}{m}\right) dx. \end{aligned} \tag{3.12}$$

Next we use $h(\lambda) = \lambda^s$, $s \in (0,1]$ and a special case of a positive function $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, to obtain a following new Hermite-Hadamard inequality for exponentially (s,m)-convex functions in the second sense.

Corollary 3.5 *Let f be a nonnegative exponentially (s,m)-convex function in the second sense on $[0,\infty)$ where $s, m \in (0,1]$. If $f \in L_1[a,b]$, where $0 \leq a < b < \infty$, then the following inequalities hold*