TWO–SIDED ESTIMATE OF THE WEYL–TYPE OPERATOR FOR $p > q$

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Dedicated to Professor Lars-Erik Persson on the occasion of his 80th anniversary

Abstract. In this paper, the necessary and sufficient conditions for the boundedness of the Weyl-type operator from the weighted space $L_{p,w} = L_{p,w}(I)$ to the weighted space $L_{q,v} = L_{q,v}(I)$ are obtained for $p > q$.

1. Introduction

Let I=(a,b), $0 \le a < b \le \infty$, $0 < \alpha < 1$ and let u and v almost everywhere be locally integrable and positive functions on the interval I. Also, let be, $1 < p < \infty$, $0 < q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

$L_{p,w}$ is a weighted Lebesgue space with the norm $\|f\|_{p,w} := (\int_a^b |f(x)|^p w(x)\,dx)^{\frac{1}{p}} < \infty$. Let us denote all functions $f : I \to \mathbb{R}$ measurable in the interval I.

Moreover, $W : I \to \mathbb{R}$ is non-negative, strictly increasing and let W be a locally absolutely continuous function on the interval, for all $x \in I$, where $\frac{dW(x)}{dx} = w(x)$. In the paper, from the space $L_{p,w} = L_{p,w}(I)$ to the space $L_{q,v} = L_{q,v}(I)$, we consider the following operator:

$$Tf(x) := \int_x^b \frac{\left(\ln \frac{W(s)}{W(s)-W(x)}\right)^{\beta} u(s)W^{\gamma}(s)f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}},$$

where $x \in I$, $0 < \alpha < 1$, $\gamma \le \beta \le 0$. Its dual position in the $\gamma = 0$ and interval (a,x) of this operator is considered in the paper [3].

If $u = 1, \beta = 0, \gamma = 0$, the independent condition of our operator T, the operator K becomes a fractional order integral operator of the function f with respect to the function W:

$$Kf(x) := \int_x^b \frac{f(s)w(s)ds}{(W(s)-W(x))^{1-\alpha}},$$

where $x \in I$. The fact that this operator K is measured from the space $L_{p,w}$ to the space $L_{q,v}$ was obtained in the scientific paper [1].

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If the operator K contains $W(x)=x$, this operator becomes the Weyl operator.

The boundedness and compactness of the Weyl operator and its dual, the Riemann–Liouville operator, for different parameters p and q and different conditions on the weight functions were proven in [2] and [4–8].

Now we assume that the function W is positive strictly increasing and locally absolutely continuous function on I and $\lim\limits_{x\to+0} W(x)=0$.

The inequality of the form $A\le cB$ is written in the form $A\ll B$, where the positive constant c may be dependent on the parameters p, q, α, β, γ and the relation $A\approx B$ means that $A\ll B\ll A$. We denote the set of all integer number by $\mathbb{Z}$, and χ_E is the characteristic function of the set of E.

2. Auxilliary statements

Also, together with the operator T, we consider Hardy type operators H that map from the space $L_{p,w}=L_{p,w}(I)$ to the space $L_{q,v}=L_{q,v}(I)$ as follows:

$$Hf(x):=W^{\beta}(x)\int_x^b u(s)W^{\gamma+\alpha-\beta-1}(s)f(s)w(s)ds,\ x\in I.$$

In addition, we consider the properties of the function $\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$, where $\ln\left(\frac{W(s)}{W(s)-W(x)}\right)=\int_0^x\frac{w(t)\,dt}{W(s)-W(t)}$, if $s>x\ge 0$. Accordingly, we have the following inequality:

$$\frac{W(x)}{W(s)-W(x)}\ge\ln\left(\frac{W(s)}{W(s)-W(x)}\right)=\int_0^x\frac{w(t)\,dt}{W(s)-W(t)}\ge\frac{W(x)}{W(s)},\ s>x>0.$$

The function $\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$ increases with respect to the variable x and decreases with respect to the variable s. Also, the functions $\frac{1}{W(x)}\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$ and $W(s)\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$ increase with respect to the variable x and decrease with respect to the variable s for $s>x>0$. Indeed:

$$\frac{\partial}{\partial s}\left(W(s)\ln\left(\frac{W(s)}{W(s)-W(x)}\right)\right)=w(s)\ln\left(\frac{W(s)}{W(s)-W(x)}\right)-\frac{w(s)W(x)}{W(s)-W(x)}<0,$$

$$\frac{\partial}{\partial x}\left(\frac{1}{W(x)}\ln\left(\frac{W(s)}{W(s)-W(x)}\right)\right)=\frac{w(x)}{W^2(x)}\left(\frac{W(x)}{W(s)-W(x)}-\ln\left(\frac{W(s)}{W(s)-W(x)}\right)\right)>0$$

for $x\in(0,s)$.

THEOREM A. *Let $0<q<p<\infty, p>1$. Then the Hardy-type operator H is* *measurable from the space $L_{p,w}$ to the space $L_{q,v}$ if and only if $B < \infty$, where:*

$$B = \left(\int_a^b \left(\int_z^b u^{p'}(s) W^{p'(\gamma+\alpha-\beta-1)}(s) w(s) ds \right)^{\frac{q(p-1)}{p-q}} \times \right.$$

$$\left. \times \left(\int_a^z W^{q\beta}(x) v(x) dx \right)^{\frac{q}{p-q}} W^{q\beta}(z) v(z) dz \right)^{\frac{p-q}{pq}}.$$

Moreover, $\|H\| \approx B$.

3. The boundedness of the operator T in the case $q < p$

The main result of this section is given in the following theorem:

THEOREM 1. (Main) $0 < q < p < \infty, 0 < \alpha < 1, \gamma \leq \beta \leq 0, p > \frac{1}{\alpha}$ *and u be a non-decreasing (positive) function in the interval I. Then the operator T is bounded from the space $L_{p,w}$ to the space $L_{q,v}$ if and only if $B < \infty$. Moreover, $\|T\| \approx B$.*

Proof. Necessity: Let the operator T be bounded from the space $L_{p,w}$ to the space $L_{q,v}$. Then, using the properties of the function $\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$ for $s > x > 0$, we obtain the inequality $\frac{1}{(W(x)-W(s))^{1-\alpha}} \geq \frac{1}{(W(x))^{1-\alpha}}$ for all $x \in I$. Accordingly, we have the inequality $Tf(x) \geq Hf(x)$ for all $x \in I$ with respect to the operator T for $f \geq 0$ and the Hardy-type operator H, where the operator H is bounded from the space $L_{p,w}$ to the space $L_{q,v}$ and becomes $\|T\| \gg \|H\|$. Therefore, by Theorem A, $B < \infty$ and $\|T\| \gg B$. Thus, the necessity is fulfilled.

Sufficiency: Let $B < \infty$. Since the function W is continuous and strictly increasing on interval I and $W(a) = 0$, then for any $k \in \mathbb{Z}$ the set $\{x : W(x) \leq 2^k, x \in I\}$ is non-empty. Denoting $x_k = \sup\{x : W(x) \leq 2^k, x \in I\}$ we obtain a sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ such that $0 < x_k \leq x_{k+1}$, $\forall k \in \mathbb{Z}$ and if $x_k < b$, then $W(x_k) = 2^k$, $2^k \leq W(x) \leq 2^{k+1}$ for $x_k \leq x \leq x_{k+1}$, $\int_{x_{k-1}}^{x_k} w(s)ds = 2^{k-1}$ and if $x_{k+1} = b$, then $\int_{x_k}^{x_{k+1}} w(s)ds \leq 2^k$. These facts will be used below without reminders. We assume that $I_k = [x_k, x_{k+1})$, $k \in \mathbb{Z}$. If $k_\infty = \inf\{k \in \mathbb{Z} : \sup_{x>0} W(x) \leq 2^k\}$, then $k+1 \leq k_\infty$ for $0 < x_k \leq x_{k+1}$. Then, without loss of generality, if $k_\infty = \infty$, then $I = \bigcup_{k\in\mathbb{Z}} I_k = \bigcup_{k\in\mathbb{Z}} [x_k, x_{k+1})$.

Accordingly, let $f \in L_{p,w}$ and $f \geq 0$, then for relations $x_k \leq x \leq x_{k+1}$, $x_{k-1} < x_k$ and $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ by the inequality:

$$\|Tf\|_{q,v}^q = \sum_k \int_{x_{k-1}}^{x_k} v(x) \left| \int_x^{x_{k+1}} \frac{\left(\ln \frac{W(s)}{W(s)-W(x)}\right)^\beta u(s) W^\gamma(s) f(s) w(s)\, ds}{(W(s)-W(x))^{1-\alpha}} \right.$$

$$\left. + \int_{x_{k+1}}^b \frac{\left(\ln \frac{W(s)}{W(s)-W(x)}\right)^\beta u(s) W^\gamma(s) f(s) w(s)\, ds}{(W(s)-W(x))^{1-\alpha}} \right|^q dx$$

$$\leq 2^{q-1}\left(\sum_k \int_{x_{k-1}}^{x_k} v(x)\left|\int_x^{x_{k+1}} \frac{\left(\ln\frac{W(s)}{W(s)-W(x)}\right)^\beta u(s)W^\gamma(s)f(s)w(s)\,ds}{(W(s)-W(x))^{1-\alpha}}\right|^q dx\right.$$

$$\left.+\sum_k \int_{x_{k-1}}^{x_k} v(x)\left|\int_{x_{k+1}}^{b} \frac{\left(\ln\frac{W(s)}{W(s)-W(x)}\right)^\beta u(s)W^\gamma(s)f(s)w(s)\,ds}{(W(s)-W(x))^{1-\alpha}}\right|^q dx\right)$$

$$= 2^{q-1}(J_1+J_2),$$

where J_1 and J_2 are the first and second terms, respectively.

Now we estimate the expressions J_1 and J_2 separately from above. First, we estimate the expression J_2, where $W(x_k)=2^k$ and taking into account the conditions $x_{k-1}\leq x\leq x_k$, we use the monotonicity of the function $\frac{1}{W(x)}\ln\left(\frac{W(s)}{W(s)-W(x)}\right)$, we make the following transformation:

$$J_2=\sum_k \int_{x_{k-1}}^{x_k} v(x)\left|\int_{x_{k+1}}^{b} \frac{\left(\frac{1}{W(x)}\ln\left(\frac{W(s)}{W(s)-W(x)}\right)\right)^\beta u(s)W^\gamma(s)W^\beta(x)f(s)w(s)\,ds}{(W(s)-W(x))^{1-\alpha}}\right|^q dx.$$

Using the arguments and computations from [1] and [3], we estimate the expression J_2 from above.

$$J_2\leq \int_a^b v(x)W^\beta(x)\left|\int_x^b u(s)W^{\gamma+\alpha-\beta-1}(s)f(s)w(s)\,ds\right|^q dx = \|Hf\|^q_{(q,v)},$$

then $\|Hf\|^q_{(q,v)} \ll B^q\|f\|^q_{(p,w)}$ according to Theorem A. Thus, $J_2\ll B^q\|f\|^q_{(p,w)}$. Similarly, following the aforementioned, and using the arguments and computations from [1] and [3], we also estimate the expression J_1 from above:

$$J_1=\sum_k \int_{x_{k-1}}^{x_k} v(x)\left|\int_x^{x_{k+1}} \frac{\left(\ln\left(\frac{W(s)}{W(s)-W(x)}\right)\right)^\beta u(s)W^\gamma(s)f(s)w(s)\,ds}{(W(s)-W(x))^{1-\alpha}}\right|^q dx \ll B^q\|f\|^q_{p,w}.$$

Then from the calculated integral $\int_a^b v(x)|Tf(x)|^q dx \leq 2^{q-1}(J_1+J_2)$ and the found inequalities J_1 and J_2, we come to the following conclusion: $\|Tf\|_{q,v}\ll B\|f\|_{p,w}$. Therefore, the operator T is bounded from the space $L_{p,w}$ to the space $L_{q,v}$ in the case $q<p$. Moreover, since $\|T\|\gg B$, then $\|T\|\ll B$. Then we have $\|T\|\approx B$ and $B<\infty$. We have shown that sufficiency is fulfilled. The theorem is fully proved. □

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PREDICTING TENSILE MODULUS WITH KNOWN LOCAL MATERIAL PROPERTIES OF FDM 3D PRINTED PARTS

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Dedicated to Professor Lars-Erik Persson on the occasion of his 80th anniversary

Abstract. Experiments were conducted using Tough polylactide (PLA) material to determine material constants, such as Young's modulus and Shear modulus, based on different printing orientations. Classical Laminate Theory (CLT) was then used to predict the tensile modulus of samples with varying raster angles, and the results were compared with experimental data. The CLT predictions showed a mean relative error of 4.26%, demonstrating its effectiveness in estimating the mechanical properties of Fused Deposition Modeling (FDM)-printed components. Future work will focus on addressing discrepancies observed in the tool-path behavior during printing and investigating the impact of layer bonding on part performance.

1. Introduction

Fused Deposition Modeling (FDM) 3D printing is a popular additive manufacturing process. Because of its ability to create complex geometries that are cost-effective and fast. Predicting the mechanical properties of FDM parts, however, is difficult. This is due to the anisotropic nature of FDM parts [3].

To address these challenges, this paper explores the application of Classical Laminate Theory (CLT) [7], a method traditionally used for composite materials, to model the mechanical behavior of FDM-printed parts. By treating each printed layer as a distinct lamina, CLT provides a framework for predicting the overall mechanical properties of printed components, taking into account the anisotropic behavior and specific printing conditions.

2. Classical Laminate Theory

To predict the mechanical behavior of an entire laminated structure under external loads, it is necessary to transform local material properties into a global coordinate system. The global coordinate system represents the overall structure and can differ significantly from the orientation of individual layers. Using transformation matrices, CLT combines the local mechanical properties of each lamina into an overall response

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for the laminate, accounting for interactions between layers with different orientations. This allows for the accurate prediction of mechanical behavior, such as stress and strain distributions, in complex multilayered materials, making CLT a beneficial tool for analyzing the anisotropic behavior of FDM 3D-printed parts.

From the governing equation for plane stress (generalized Hooke's Law), we have

$$\sigma_{12} = Q\varepsilon_{12}, \tag{1}$$

where the matrix Q is the stiffness matrix in local coordinates, σ and ε are the stress and strains, respectively.

(1) equates the principal stresses in the direction of the layered filament in the local coordinate system. Transforming (1) into the global coordinate system can be done using transformation. Generally, we seek to obtain

$$\sigma_{xy} = \overline{Q}\varepsilon_{xy},$$

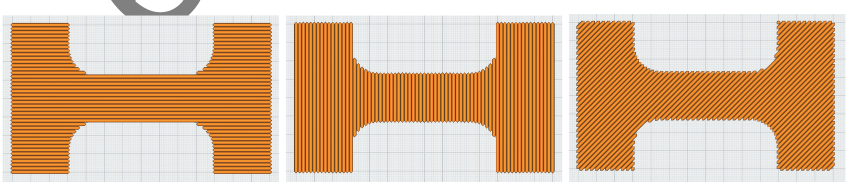
where $\overline{Q}$ is Q transformed to the global coordinate system via

$$\overline{Q} = T^{-1}QRTR^{-1},$$

where T is the transformation matrix and R is the engineering strain factorization matrix [7]. From $\overline{Q}$ the effective Young's modulus E_x can be calculated using determinants and matrices formulas given by [7].

3. Experiment

Local material properties must be known to estimate the effective tensile modulus E_x for a specimen using CLT. Specimens are printed using Tough PLA followed by the procedure used in [4]. To initialize the CLT program, each material constant (E_1, E_2, and G_{12}) must be accounted for. E_1, and E_2 are determined by printing a specimen in the longitudinal and transverse direction, respectively, as shown in Figure 1(a) and Figure 1(b).



(a) Longitudinal E_1. (b) Transverse E_2. (c) Sequence of $[\pm 45]_R$ E_{45}.

Figure 1: Different raster angles to determine the material constants E_1, E_2 and E_{45}.

To determine the shear modulus G_{12}, the printing sequence is set to a $[\pm 45]_R$ orientation, which yields the modulus E_{45}, as shown in Figure 1(c). According to [5], G_{12} can be calculated from E_1, E_2, E_{45} and ν_{12}, where ν_{12} is set to 0.36 from [2].

The initial experiments showed that the local material constants were $E_1 = 1008.621$ MPa, $E_2 = 711.108$ MPa, and that $E_{45} = 740.300$ MPa, which resulted in a Shear modulus $G_{12} = 268.864$ MPa.

Here CLT is used to estimate the Tensile modulus of a specimen with various raster angles $[0, \pm 30, \pm 60, 0, 90]_R$. Results from tensile testing are given in Table 1.

Property	**Specimen 1**	**Specimen 2**	**Specimen 3**
Measured Tensile Modulus, E (MPa)	891.902	888.568	843.227
Mean values			
Mean Measured Tensile Modulus, E (MPa)		874.566	
CLT Predicted Tensile Modulus, E_{CLT} (MPa)		837.340	
Error			
Mean Absolute Error (MPa)		37.226	
Mean Relative Error (%)		4.26	

Table 1: Comparison of CLT predictions and experimental tensile results

4. Conclusion

The predicted Tensile modulus from CLT was $E_{CLT} = 837.340$ MPa. To evaluate CLT computation, three specimens with line patterns and raster angles of $[0, \pm 30, \pm 60, 0, 90]_R$ were printed and tested, yielding a mean Tensile modulus of 874.566 MPa. This result had a relative error of 4.26% to the predicted modulus from CLT, suggesting that CLT is a reliable mathematical model for determining the mechanical properties of FDM 3D-printed parts.

5. Future Work

During the experiments, the effect of concave geometries on the FDM tool path was observed, specifically the jumping between sections during printing of specimens, see Figure 2. Observations showed that the tool path irregularities occurred at the same location during the printing process. How this phenomenon affects the strength of the printed part is of great interest in the future. Here, as in other parts of our work, we are inspired to do the mathematical modeling by using modern ideas in convexity theory, see [6].

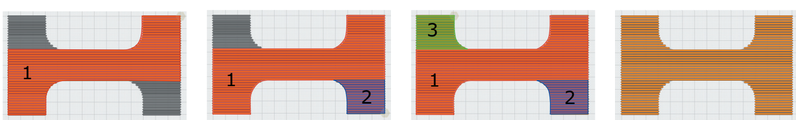


Figure 2: Tool-path for FDM 3D-printer for line pattern with raster angle 0°.

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LOCAL APPROXIMATION PROBLEMS AND KOROVKIN–TYPE THEOREMS IN FUNCTION SPACES: A SHORT SURVEY

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Dedicated to Professor Lars-Erik Persson with deep respect and friendship

Abstract. The paper is devoted to present a short survey on some very recent results concerning local approximation problems by positive linear operators on function spaces. The implemented methods are typical of those of the Korovkin-type approximation theory. Some applications and open problems are illustrated as well.

1. Introduction and statement of the problem

Very recently, by implementing well-established methods arising from the Korovkin-type approximation theory ([1], [6]), in a series of papers ([2] - [5]) we investigate local approximation problems by positive linear operators on function spaces.

The starting point is the following result due to P. P. Korovkin (see, e.g., [9]):

THEOREM. *Given a real interval I, consider a linear subspace E of real-valued functions on I containing the functions $\mathbf{1}$, $e_1(t) := t$ and $e_2(t) := t^2$ $(t \in I)$, and a sequence $(L_n)_{n\geq 1}$ of positive linear operators on E.*
Given a compact subinterval K of I, if for every $h = \mathbf{1}, e_1, e_2$,

$$\lim_{n\to\infty} L_n(h) = h \text{ uniformly on } K,$$

then

$$\lim_{n\to\infty} L_n(f) = f \text{ uniformly on } K$$

for every bounded function $f \in E$ which is continuous on each point of K.

The most renowned theorem of Korovkin is a special case of the result above and it concerns the particular setting where

$$I \text{ is compact, } E = C(I) \text{ and } K = I.$$

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